Computing Conformal Maps onto Circular Domains

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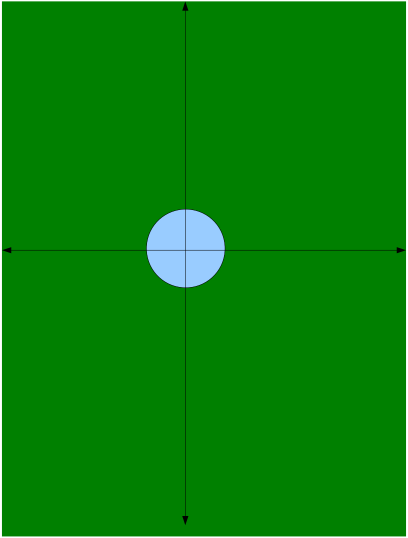
Outline

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. As usual, topologize this with the basis consisting of all open rational disks as well as all sets of the form



Definition

1. A *domain* is an open connected subset of $\hat{\mathbb{C}}$.
2. A domain is n-connected if its complement has exactly n components.
3. A domain is *degenerate* if a component of its complement consists of a single point.
4. A domain is *circular* if every component of its complement is a closed disk.

Let $\mathbb{D}$ be the unit disk centered at the origin.

Example

- $\mathbb{D}$ is 1-connected. So is $\hat{\mathbb{C}} - \overline{\mathbb{D}}$.
- Every annulus is 2-connected.

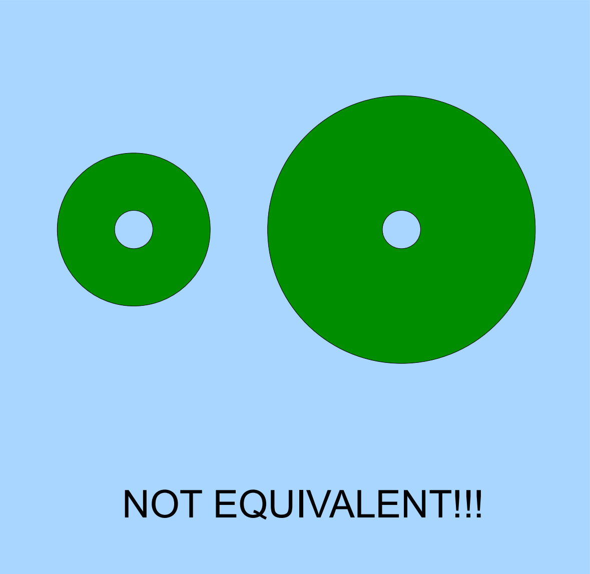
Theorem (Riemann Mapping Theorem)

Every non-degenerate 1-connected domain is conformally equivalent to $\mathbb{D}$.

Question

Are all non-degenerate n-connected domains conformally equivalent?

Answer $=$ NO!


NOT EQUIVALENT!!!

This leads to the discussion of *canonical domain classes.*

Definition

A *canonical domain class* is a set of finitely connected domains $\mathcal{S}$ such that each finitely connected non-degenerate domain is conformally equivalent to a domain in $\mathcal{S}$.

There are as many of these as there are grains of sand...

- Slit domains
  - Slit disk domains
  - Slit annulus domains
  - Circular slit domains
  - Parallel slit domains
  - Radial slit domains
- Circular domains
- Polygonal domains
- Polyarc domains
- *etc.*

Some problems from Pour-El and Richards:

1. What is the effective content of the Riemann Mapping Theorem?
2. What is the effective content of conformal mapping for multiply connected domains?

Some related work:

- Schwarz-Christoffel formula for mapping $\mathbb{D}$ onto polygonal domains.
- Schiffer, 1950: constructed slit domain maps from the Green's function for the input domain.
- Bishop 1967, Bishop and Bridges 1985: constructive proof of the Riemann mapping theorem.
- Hertling, 1999: for every non-degenerate 1-connected domain, D, a conformal map f of D onto $\mathbb{D}$ can be uniformly computed from D and its boundary. This means:

there is a Turing machine that given

- a list of all basic sets whose closures are contained in a non-degenerate 1-connected domain D,
- a list of all basic sets which hit ∂D

enumerates all pairs (S_1, S_2) such that

- S_1, S_2 are basic
- $\overline{S_1} \subseteq D$
- $f[\overline{S_1}] \subseteq S_2$.

Recent importance of circular domains:

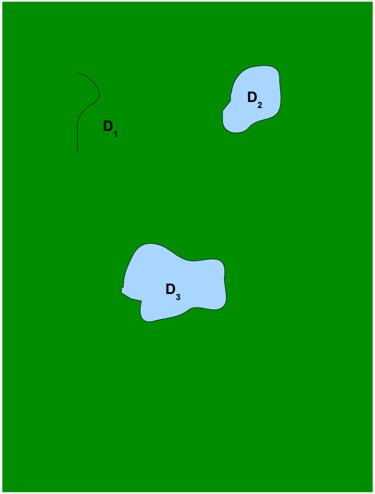
- Explicit formulas for Green's functions
- Applications to aircraft design
- Heat-transfer equations
- Canonical domains in recent research in conformal mapping onto polygonal domains

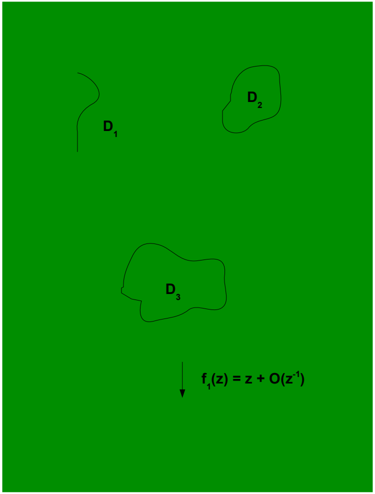
Every non-degenerate n-connected domain is conformally equivalent to infinitely many circular domains. However, if we normalize the map we get uniqueness:

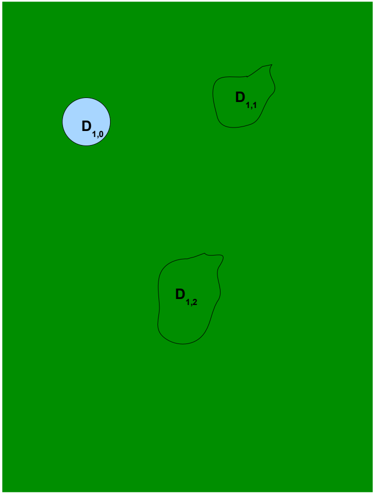
Theorem (Koebe [7])

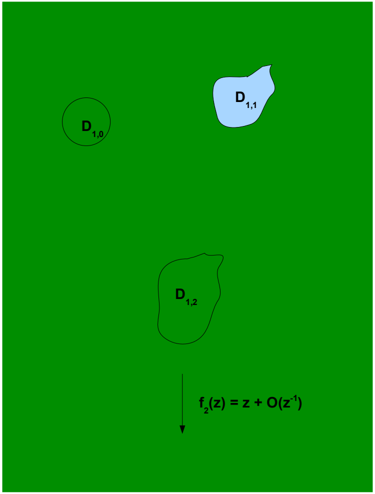
For every non-degenerate n-connected domain D that contains ∞, there is a unique circular domain C_D and a unique conformal map f_D of D onto C_D such that $f_D(z) = z + O(z^{-1})$.

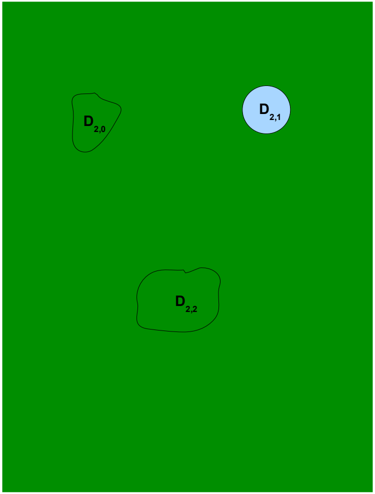
It follows from Hertling's theorem that when $n = 1$, $(D, \partial D) \mapsto (f_D, \partial C_D)$ is computable.

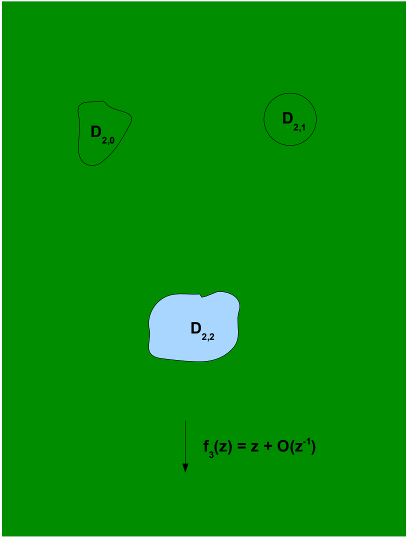

D1
D2
D3

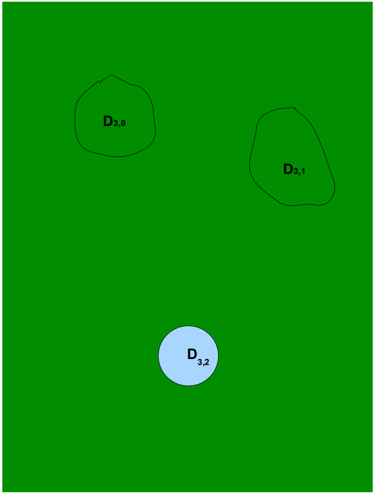

D1
D2
D3
f1(z) = z + O(z-1)

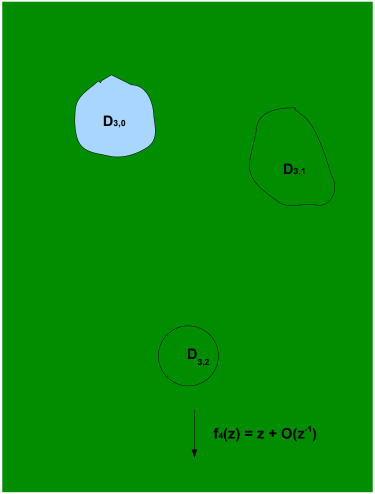

D 1,0
D 1,1
D 1,2


D 1,0
D 1,1
D 1,2
f2(z) = z + O(z-1)

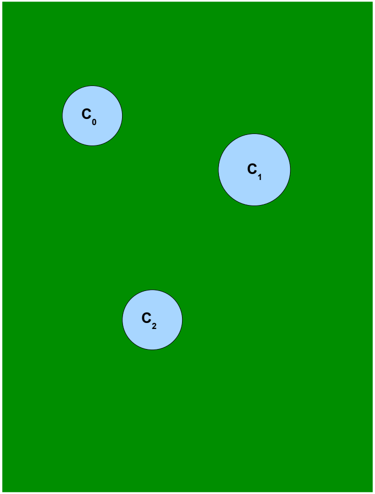

D 2,0
D 2,1
D 2,2


D2,0
D2,1
D2,2
f3(z) = z + O(z-1)


D3,0
D3,1
D3,2


D3,0
D3,1
D3,2
f4(z) = z + O(z-1)

... after infinitely many iterations ...


C0
C1
C2

Let

$$\begin{aligned} g_1 &= f_1 \\ g_{k+1} &= f_{k+1} \circ f_k \ldots \circ f_1 \end{aligned}$$

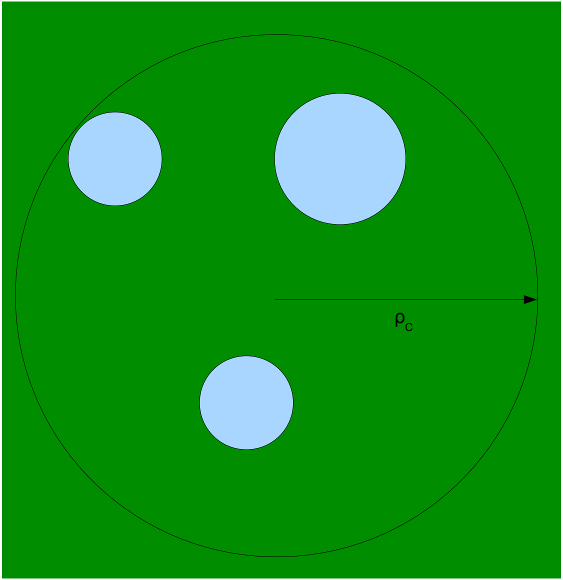
Then, $g =_{df} \lim_{k\to\infty} g_k$ exists and is a conformal mapping of D onto C_D of the form $z + O(z^{-1})$. *n.b.* Known proofs of this use fact that f_D exists!

Throughout the rest of this talk, let D range over finitely-connected non-degenerate domains only.

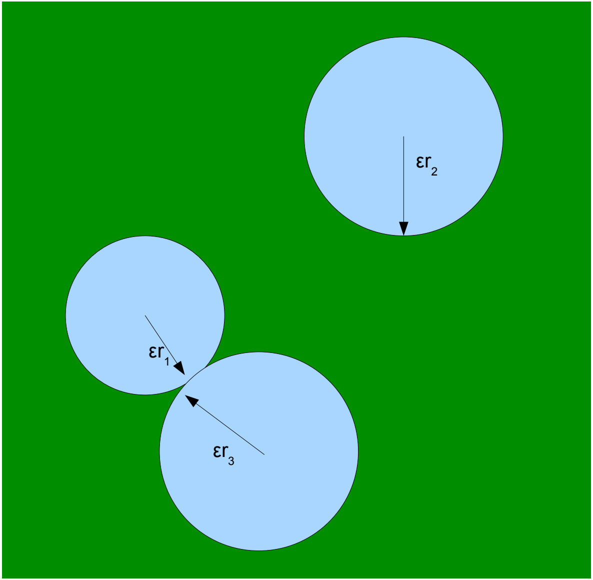
Theorem (Main result)

$(D, \partial D, n) \mapsto (f_D, C_D, \partial C_D)$ *is computable.*

Proof comes down to computing the error in the Koebe construction *from D and ∂D* So, let's sketch how to do it.

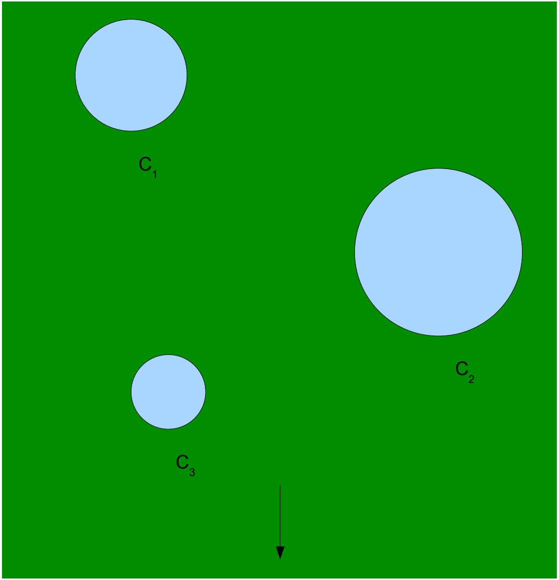

ρc

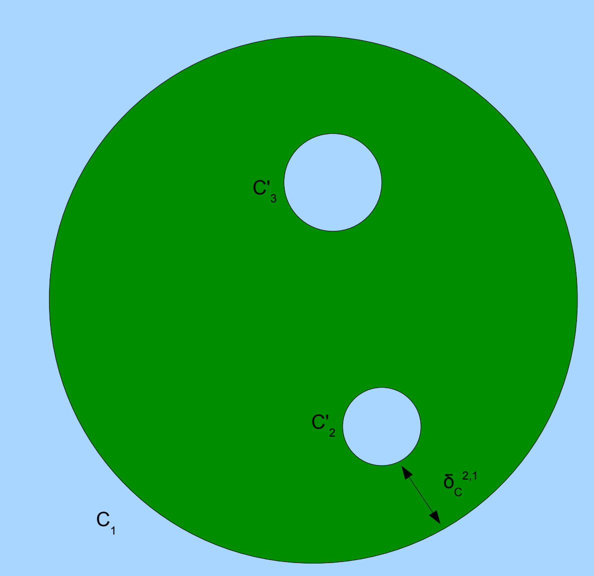
Can transfer this back to any D by setting $\rho_D = \rho_{C_D}$.


εr2
εr1
εr3

$\mu_C = (\text{least } \epsilon \text{ such that two circles touch})^{-1}$.

Let $\mu_D = \mu_{C_D}$.


C_1
C_2
C_3


C'3
C'2
δC2,1
C1

Let $\delta_C = \min_{i,j} \delta_C^{i,j}$.

Again, we can transfer this to any domain D by setting Let $\delta_D = \delta_{C_D}$.

Notation

Let

$$\gamma_D = \frac{2\rho_D^2}{\pi\delta_D}\left[\frac{2[\pi\mu_D^{-1}]^2}{\ln \mu_D^{-1}} + 1\right].$$

Theorem (Henrici, 1986 [5])

For all $z \in D - \{\infty\}$ *and all* $j \in \mathbb{N}$, $|g_j(z) - g(z)| \leq \gamma\mu^{4\lfloor j/n \rfloor}$.

Key point: Proof assumes f_D and C_D exist! So, it does NOT provide a constructive proof of the existence of f_D. All proofs of the convergence of the Koebe construction possess this feature!

To use this result we need to:

- Compute an upper bound on ρ_D.
- Estimate μ_D from above but below 1.
- Estimate δ_D from below.

Proposition

From $(D, \partial D, n)$, we can compute an upper bound on ρ_D.

Notation

- *Let $r_{min}(D)$ be the smallest radius of a circle in C_D.*
- *Let $d_{min}(D)$ be the smallest distance between two circles in C_D.*

Proposition

$$\delta_D \geq \frac{1}{\frac{1}{r_{min}(D)} + \frac{1}{d_{min}(D)}}.$$

So, to estimate δ_D from below, we only need to estimate $r_{min}(D)$ and $d_{min}(D)$ from below.

Theorem

We can compute a positive lower bound on $r_{min}(D)$ from $(D, \partial D, n)$.

Proof uses recent results on:

- Distortion of capacity (Thurman, 1994, [9]),
- Computation of capacity (Ransford and Rostand, 2007 [8])

Theorem

We can compute a positive lower bound on $d_{min}(D)$ given $(D, \partial D, n)$.

Proof uses a generalization of the Schwarz-Pick Lemma proved in 1993 by He and Schramm [3]

Theorem

From $(D, \partial D, n)$ can compute a number between μ_D and 1.

Follows from results just quoted. It now follows we can compute suitable bound on error in Koebe construction.

Corollary

Given a smooth Jordan domain D and the derivatives of its boundary curves, one can compute the homeomorphic extension of f_D to $\overline{D}$.

n.b. We still do not have a constructive existence proof!

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