EFFECTIVE VERSIONS OF LOCAL CONNECTIVITY PROPERTIES

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Abstract. We investigate, and prove equivalent, effective versions of local connectivity and uniformly local arcwise connectivity for connected and computably compact subspaces of Euclidean space.

1. Introduction

Computability theory is concerned with the theoretical and practical limitations of discrete computing devices as well as the effective content of mathematical theorems. That is, when is the solution operator for a given class of problems amenable to computation by a discrete computing device or susceptible to an explicitly constructive description? Such a theory requires precise mathematical foundations in order to achieve rigorous demonstration of its results. For computation with discrete data, such as the natural or rational numbers, the foundations laid by the work of Turing, Church, and Kleene suffice. The interested reader may find a historical survey of the genesis of these ideas in [6] and detailed developments in standard texts such as [4]. These notions are also sufficient for the exploration of the effective content of theorems in algebra. See, *e.g.* [5].

When one wants to consider the theorems of analysis and topology, it is essential however to have a sound theory of computation with continuous data. Such a theory should extend without overriding the framework for discrete data. In addition, in it the fundamental mathematical notion of approximation should bridge the divide between the continuous and the discrete. Several such theories are available. For example, see [1],[10], [11], [15]. We will base our work here on the Type Two Effectivity approach to computable analysis as developed in [16]. However, many of our results could be translated into the framework of other approaches.

In order to investigate the effective content of a theory it is first necessary to formulate effective versions of its basic definitions. Roughly speaking, an effective version of a property insists that we can actually compute from any entity for which the property holds all objects whose existence is thusly entailed. Fundamental to topology and much of analysis are the notions of compact, closed, open, and connected set. Effective versions of compact, closed, and open sets within the framework of Type Two Effectivity were explored in detail by V. Brattka and K. Weihrauch in [3]. Local connectivity perhaps sits on a lower tier than these first four topological concepts, but nevertheless has played an important role in the development of topology and analysis. A space is locally connected if each of its points has a local basis of connected sets. This property plays a crucial role in the

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characterization of space-filling curves. *i.e.*, the Hahn-Mazurkiewicz Theorem [8], [12]. The notion of effective local connectivity first appeared in J. Miller's paper on effective embeddings of balls and spheres [13]. More recently, it has been used by V. Brattka in connection with computation of functions from their graphs [2].

Here, we consider two effective versions of a cousin of local connectivity: uniform local arcwise connectivity. Roughly speaking, a space is uniformly locally arcwise connected if all points sufficiently close in the space can be joined by an arc of arbitrarily small diameter. Here, an *arc* is a compact, connected set for which there are exactly two points with the property that the removal of either one of them from the set results in another connected set. There are at least two ways to create an effective version of this notion. On the one hand, we may want to compute how close two points need to be in order to join them by an arc of diameter below some given value. On the other hand, we may also want to compute such an arc. These simple observations lead to the notions of effective uniform local arcwise connectivity and strongly effective uniform local arcwise connectivity. These are defined precisely in Section 3.

Our main result is that on subsets of Euclidean space that are connected and computably compact, the notions of effective local connectivity, effective uniform local connectivity, and strongly effective uniform local connectivity are equivalent. Roughly speaking, a subset of $\mathbb{R}^2$ is computably compact if it can be plotted with arbitrary precision by a discrete computing device. A precise definition which also covers spaces of dimension greater than two is given in Section 3. See also [2].

One interpretation of this result is that effective local connectivity provides, for spaces which are computably compact, the precise amount of information necessary for the computation of arcs between points in the space. Another interpretation is that it provides an effective version of a classical result: every Peano continuum is uniformly arcwise connected (see, *e.g.* [9]). By a Peano continuum is meant a space which is compact, metrizable, connected, and locally connected.

2. Summary of pertinent notions and results from topology

The material in this section is taken from Hocking and Young [9] and Munkres [14]. All spaces considered are subspaces of $\mathbb{R}^n$.

Let d be the Euclidean metric on $\mathbb{R}^n$. If $X \subseteq \mathbb{R}^n$ is bounded, then we let

$$\operatorname{diam}(X) = \sup\{d(x,y) \mid x, y \in X\}.$$

If $X, Y \subseteq \mathbb{R}^n$ are closed, then we let

$$d(X,Y) = \min\{d(x,y) \mid x \in X \ \wedge \ y \in Y\}.$$

Let $B_\epsilon(p)$ denote the open ball of center p and radius ϵ. When, $S \subseteq \mathbb{R}^n$, we let

$$B_\epsilon(S) = \bigcup_{p \in S} B_\epsilon(p).$$

A *path* is a continuous image of $[0,1]$. An *arc* is a homeomorphic image of $[0,1]$. Such a homeomorphism is called a *parameterization* of the arc. If A is an arc, then there are exactly two points in A such that the removal of either one of these points from A yields a connected set. We call these points the *endpoints of* A. It follows that if f is a parameterization of an arc A, then $f(0)$ and $f(1)$ are the endpoints of A. If x, y are the endpoints of an arc A, then we say that A is *an arc from* x *to* y.

A topological space X is *arcwise connected* if for every distinct $x, y \in X$ there is an arc in X from x to y. On the other hand, X is *pathwise connected* if for every $x, y \in X$ there is a continuous function $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. It follows from the theorems discussed below that every pathwise connected space is arcwise connected.

Suppose X is a topological space, and that $a, b \in X$. Write $a \sim b$ if there is a connected set C that contains a and b. It follows that $\sim$ is an equivalence relation. Its equivalence classes are called the *connected components* of X. We define *path component* similarly by using path connected sets.

We now discuss local connectivity properties.

Definition 2.1. A topological space X is *locally connected (LC)* if for every $p \in X$ and every neighborhood of p, U, there is a connected neighborhood of p, V, such that $V \subseteq U$.

The most pertinent results about locally connected spaces are the following. Proofs can be found in Section 3-4 of [14].

Theorem 2.2. *Let X be a topological space.*

1. *X is locally connected if and only if for every open $U \subseteq X$, each connected component of U is open in X.*
2. *If X is locally connected, then the connected components of X are precisely the path components of X.*

Definition 2.3. A metric space (X, d) is *uniformly locally arcwise connected (ULAC)* if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X$ with $d(x, y) < \delta$, there exists an arc A in X from x to y whose diameter is less than ϵ.

A *continuum* is a compact, connected, and metrizable topological space. A *Peano continuum* is a locally connected continuum. The most pertinent results about Peano continua are the following. Proofs can be found in [9].

Theorem 2.4 (**Hahn-Mazurkiewicz Theorem,** [8]**,** [12]**).** *Suppose P is a subspace of a Hausdorff space X. Then, P is a Peano continua if and only if P is the image of* $[0,1]$ *under a continuous map.*

Theorem 2.5. *Every Peano continuum is arcwise connected.*

It now follows that every pathwise connected subset of Euclidean space is arcwise connected.

By a simple application of the Lebesgue Number Theorem, one can now prove the following.

Theorem 2.6. *Every Peano continuum is uniformly locally arcwise connected.*

Theorem 2.5 is proven by means of *simple chains.* These will be a valuable tool for us as well. We define them here.

Definition 2.7. Let $(U_1, \ldots, U_k)$ be a sequence of open sets. Then, $(U_1, \ldots, U_k)$ is a *simple chain* if $U_i \cap U_j \neq \emptyset$ precisely when $|i - j| \leq 1$.

We define some associated terminology.

Definition 2.8. Suppose $(U_1, \ldots, U_k)$ is a simple chain. If

$$x \in U_1 - \bigcup_{j=2}^{k} U_j,$$

and if

$$y \in U_k - \bigcup_{j=1}^{k-1} U_j,$$

then we say that $(U_1, \ldots, U_k)$ is a simple chain *from x to y*.

Definition 2.9. Suppose $(U_1, \ldots, U_k)$ is a simple chain in a subspace of $\mathbb{R}^n$ and that each U_j is bounded. The *diameter* of $(U_1, \ldots, U_k)$ is the maximum of $\operatorname{diam}(U_1)$, ..., $\operatorname{diam}(U_n)$.

Definition 2.10. Suppose $(U_1, \ldots, U_k)$ and $(V_1, \ldots, V_l)$ are simple chains. We say that $(U_1, \ldots, U_k)$ *refines* $(V_1, \ldots, V_l)$ if each U_j is contained in at least one of $V_1, \ldots, V_l$.

Definition 2.11. We say that a simple chain $(U_1, \ldots, U_k)$ *goes straight through* a simple chain $(V_1, \ldots, V_l)$ if $(U_1, \ldots, U_k)$ refines $(V_1, \ldots, V_l)$ and whenever $U_i, U_j \subseteq V_s$, then $U_t \subseteq V_s$ whenever t is between i and j.

The pertinent facts about simple chains are the following. Proofs can be found in Sections 3.1 and 3.2 of [9].

Theorem 2.12. *If $\{U_\alpha\}_{\alpha \in I}$ is a covering of a connected space X by open sets, and if $x, y \in X$, then there exist $\alpha_1, \ldots, \alpha_k \in I$ such that $(U_{\alpha_1}, \ldots, U_{\alpha_k})$ is a simple chain from x to y.*

Theorem 2.13. *Suppose X is a locally connected and connected Hausdorff space and that $(U_1, \ldots, U_n)$ is a simple chain of connected open sets from a to b. Suppose $\mathcal{V}$ is a collection of open sets such that each U_i is a union of elements of $\mathcal{V}$. Then, there is a simple chain of elements of $\mathcal{V}$ from a to b that goes straight through $(U_1, \ldots, U_n)$.*

3. Background from computable analysis and computable topology

A *rational interval* is an open interval whose endpoints are rational numbers.

An *n-dimensional rational rectangle* is a set of the form $(a_1, b_1) \times \ldots \times (a_n, b_n)$ where $a_1, b_1, \ldots, a_n, b_n \in \mathbb{Q}$ and $a_i < b_i$ for all i. Let I^n be a standard computable notation for the set of all n-dimensional rational rectangles.

We will use the following naming systems only.

1. ρ^n for $\mathbb{R}^n$. Intuitively, a ρ^n-name for a point $x \in \mathbb{R}^n$ is a list of all rational rectangles to which x belongs.
2. κ_{mc} for the compact subsets of $\mathbb{R}^n$. Intuitively, a κ_{mc}-name of a compact $X \subseteq \mathbb{R}^n$ is a list of all finite covers of X by rational rectangles *each of which contains at least one point of X*. As in Section 5.2 of [16], such a covering will be called *minimal*.
3. δ_{CO} for $C([0,1], \mathbb{R}^n)$. Intuitively, a δ_{CO}-name of a continuous function $f : [0,1] \to \mathbb{R}^n$ is a list of all pairs of the form (I, R) such that I is a closed rational interval, R is an n-dimensional rational rectangle, and $f[I] \subseteq R$.

Since these are the only naming systems we will use, we will suppress their mention when discussing the computability of objects and functions.

We will make frequent use of the following principle: computation of maxima and minima of continuous functions on compact sets is computable. See, *e.g.*, Corollary 6.2.5 of [16].

Throughout the rest of this paper, $X \subseteq \mathbb{R}^n$ is a fixed continuum that is computable as a compact set.

For each $z \in X$ and each open $U \subseteq \mathbb{R}^n$ such that $z \in U \cap X$, let $C_z(U)$ denote the connected component of z in $U \cap X$. Before proceeding further, we make a small observation about these components.

Proposition 3.1. *If U, V are open subsets of $\mathbb{R}^n$ such that $z \in U \cap V \cap X$, and if $U \subseteq V$, then $C_z(U) \subseteq C_z(V)$.*

Proof. Let $C = C_z(U)$. Hence, C is a connected subset of $U \cap X$. Since $U \cap X$ is a subspace of $V \cap X$, C is a connected set in $V \cap X$. Since $z \in C$, it follows that $C \subseteq C_z(V)$. □

The following is due to J. Miller [13].

Definition 3.2. X is *effectively locally connected (ELC)* if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every $p \in X$, $X \cap B_{2^{-f(k)}}(p) \subseteq C_z(B_{2^{-k}}(p)) \subseteq B_{2^{-k}}(p)$.

A more general version of this definition is given in [2].

We propose two effective versions of the notion of a ULAC space. The second is, *prima facie*, a strengthening of the first. We will later show they are equivalent.

Definition 3.3. X is *effectively uniformly locally arcwise connected (EULAC)* if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and all $x, y \in X$, if $d(x, y) < 2^{-f(k)}$, then there is an arc in X from x to y of diameter at most 2^{-k}.

In addition to knowing there there *is* an arc between two points of sufficiently small diameter, we may also want to compute a parameterization of such an arc. This leads to the following.

Definition 3.4. X is *strongly effectively uniformly locally arcwise connected (SEULAC)* if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ that witnesses X is EULAC and it is possible to compute from names of $x, y \in X$ such that $d(x, y) < 2^{-f(k)}$ a name of a parametrization of an arc from x to y in X.

Our main result is that all three properties ELC, EULAC, and SEULAC are equivalent for computable Euclidean continua.

4. Preliminary lemmas

Lemma 4.1. *There is a computable $G :\subseteq \Sigma^* \to X$ such that for all $w \in \Sigma^*$ for which $I^n(w) \cap X \neq \emptyset$, $G(w)$ is defined and is in $X \cap I^n(w)$.*

Proof. Let p be a computable name of X. For each $w \in \Sigma^*$, we inductively define a sequence $w_0, w_1, \ldots$ as follows. Let $w_0 = w$. Once w_t has been defined, we define w_{t+1} as follows. First, let v be the smallest prefix of p such that there exists w', u such that $\iota(u) \lhd v$, $\iota(w') \lhd u$, $\mathrm{diam}(I^n(w')) < 2^{-t}$, and $\overline{I^n(w')} \subseteq I^n(w_t)$. We then define w_{t+1} to be the lexicographically least such w'. If $I^n(w) \cap X \neq \emptyset$, then w_s is defined for all s and can be computed from w and s. In this case, we define $G(w)$ to be the unique point in $\bigcap_s I^n(w_s)$.

To show that G is computable, we define a type-two machine M as follows. Given $w \in \Sigma^*$ as input, read p left to right while cycling through all numbers in $\mathbb{N}$ and all words in Σ^*. Whenever s, v are found such that w_s is defined and $I^n(w_s) \subseteq I^n(v)$, write $\iota(v)$ on the output tape. If $I^n(w) \cap X \neq \emptyset$, then it follows that $f_M(w)$ is defined and is a ρ^n-name of $G(w)$. □

Recall that a Lebesgue number for a covering $\{U_\alpha\}_{\alpha\in I}$ of a compact set C is a number $\delta > 0$ such that whenever $x, y \in C$ and $d(x, y) < \delta$, there exists $\alpha \in I$ such that $x, y \in U_\alpha$. The Lebesgue Number Lemma asserts that every open cover of a compact set has a Lebesgue number.

Lemma 4.2 (**Computable Lebesgue Number Lemma**). *There is a computable* $L :\subseteq \Sigma^* \to \mathbb{N}$ *such that for all* $w \in \Sigma^*$ *for which*

$$\{I^n(u) \mid \iota(u) \lhd w\}$$

is a covering of X*,* $L(w)$ *is defined and* $2^{-L(w)}$ *is a Lebesgue number for this covering.*

Proof. We can assume that X contains at least two points. For, otherwise we can choose $L(w)$ arbitrarily.

Let $u_1, \ldots, u_m$ be all words u such that $\iota(u) \lhd w$.

Since X is computable, we can compute $v_1, \ldots, v_l$ such that

1. $X \subseteq \bigcup_{j=1}^{l} I^n(v_j)$,
2. $X \cap I^n(v_j) \neq \emptyset$ for each j,
3. each $I^n(v_j)$ is contained in some $I^n(u_i)$, and
4. $\operatorname{diam}(I^n(v_j)) < \operatorname{diam}(X)$ for each j.

Note that any Lebesgue number for $\{I^n(v_1), \ldots, I^n(v_l)\}$ is a Lebesgue number for $\{I^n(u_1), \ldots, I^n(u_m)\}$. In addition, because of (4), there exist two points in X such that no $I^n(v_j)$ contains both of them.

Compute $w_1, \ldots, w_k$ such that

1. $X \subseteq \bigcup_{j=1}^{k} I^n(w_j)$,
2. $I^n(w_j) \cap X \neq \emptyset$ for each $j = 1, \ldots, k$, and
3. each $\overline{I^n(w_j)}$ is contained in some $I^n(v_i)$.

Let

$$C = \bigcup_{j=1}^{k} \overline{I^n(w_j)}.$$

Hence, $X \subseteq C$. At the same time $\{I^n(v_1), \ldots, I^n(v_l)\}$ is a covering of C. Note that any Lebesgue number of this covering *with respect to* C is a Lebesgue number of it as a covering of X.

Let D consist of all pairs $(p, q) \in C \times C$ for which there is no i such that $I^n(v_i)$ contains both p and q. Thus, $D \neq \emptyset$. We can write D as

$$D = \bigcap_{j=1}^{l} \left[\left((C - I^n(v_j)) \times C \right) \ \cup \ \left(C \times (C - I^n(v_j) \right) \right].$$

It follows that D is a computably compact subset of $\mathbb{R}^{2n}$. We can thus compute

$$\delta =_{df} \min\{d(p, q) \mid (p, q) \in D\}.$$

We claim that $\delta > 0$. For, suppose $\delta = 0$. Since $\{I^n(v_1), \ldots, I^n(v_l)\}$ is an open covering of C, by the Lebesgue Number Theorem there is a number $\delta_0 > 0$ such that any two points in C are contained in one of $I^n(v_1), \ldots, I^n(v_l)$ whenever the distance between them is less than δ_0. Since $\delta = 0$, there exist $(p, q) \in D$ such that

$d(p,q) < \delta_0$. However, by definition, $D \subseteq C \times C$. Hence, $p, q \in C$. Thus, there exists j such that $p, q \in I^n(v_j)$. But, it then follows by definition that $(p,q) \notin D$- a contradiction. Hence, $\delta > 0$.

Finally, since we can compute δ, we can compute $L(w)$ such that $2^{-L(w)} < \delta$. It follows that $2^{-L(w)}$ is a Lebesgue number for

$$\{I^n(v_1), \dots, I^n(v_l)\}$$

as a covering of C. □

We note that the proof of Lemma 4.2 is not uniform in that it branches on the case of whether X contains a single point and this can not be uniformly computed from a name of X. We will address the issue of uniformity in more depth in Section 8. For now, we shall focus on proving that for computable continua, ELC and EULAC are equivalent.

5. ELC and EULAC spaces

In this section, we show that X is ELC if and only if it is EULAC.

Theorem 5.1. *If X is ELC, then X is EULAC.*

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ witness that X is ELC. We can assume f is increasing.

Fix $m \in \mathbb{N}$. Since X is computably compact, we can compute $w_1, \dots, w_k$ such that $\{I^n(w_1), \dots, I^n(w_k)\}$ is a minimal cover of X with the additional property that the diameter of each $I^n(w_j)$ is smaller than $2^{-f(m+2)}$. It follows from the last inequality that $\overline{I^n(w_i)} \subseteq B_{2^{-f(m+2)}}(y)$ for all $y \in \overline{I^n(w_i)}$. We claim that, for each i, there is a connected subset of X, X_i, such that $I^n(w_i) \cap X \subseteq X_i$ and $\operatorname{diam}(X_i) \leq 2^{-m}$. To see this, for each $y \in I^n(w_i)$, let $X_{i,y} = C_y(B_{2^{-(m+2)}}(y))$. Hence, for each $y \in I^n(w_i)$,

$$\overline{I^n(w_i)} \cap X \subseteq B_{2^{-f(m+2)}}(y) \cap X \subseteq X_{i,y} \subseteq B_{2^{-(m+2)}}(y).$$

Also, since X is locally connected, $X_{i,y}$ is open in X and path connected. Hence, $X_{i,y}$ is arcwise connected. We then let

$$X_i = \bigcup_{y \in I^n(w_i) \cap X} X_{i,y}.$$

Hence, $X_i \supseteq \overline{I^n(w_i)} \cap X$. Suppose $z_0, z_1 \in X_i$. Then, there exist $y_0, y_1 \in I^n(w_i) \cap X$ such that $z_j \in X_{i,y_j}$ for each j. Since $\overline{I^n(w_i)} \cap X \subseteq X_{i,y}$, it follows that $y_1 \in X_{i,y_0}$. Hence, there is an arc in X_{i,y_0} from z_0 to y_1, and an arc in X_{i,y_1} from y_1 to z_1. It follows that X_i is path connected and hence arcwise connected. Since

$$X_{i,y} \subseteq B_{2^{-(m+2)}}(y) \subseteq B_{2^{-(m+2)}}(I^n(w_i))$$

for each $y \in I^n(w_i) \cap X$, and since

$$\operatorname{diam}(I^n(w_i)) < 2^{-f(m+2)} < 2^{-(m+2)},$$

it also follows that $\operatorname{diam}(X_i) < 2^{-m}$.

By Lemma 4.2, we can compute $g(m) \in \mathbb{N}$ so that $2^{-g(m)}$ is a Lebesgue number for the covering

$$\{I^n(w_1), \dots, I^n(w_k)\}.$$

It follows that if $d(x,y) < 2^{-g(m)}$, then there exists i such that $x, y \in I^n(w_i)$ and so x, y are connected by an arc in X of diameter at most 2^{-m}. □

We now prove the converse of Theorem 5.1 holds for computable Euclidean continua.

Theorem 5.2. *If X is EULAC, then X is ELC.*

Proof. Let g witness that X is effectively uniformly locally arcwise connected. We claim that g witnesses that X is effectively locally connected. For, let $n \in \mathbb{N}$, and let $y \in X$. Let $C = C_y(B_{2^{-n}}(y))$. We claim that $X \cap B_{2^{-g(n)}}(y) \subseteq C$. For, let $z \in B_{2^{-g(n)}}(y)$. Then, there is an arc from y to z in X of diameter less than 2^{-n}. If this arc were to contain a point not in $B_{2^{-n}}(y)$, then its diameter would be at least as large as 2^{-n}. Hence, this arc is entirely contained in $B_{2^{-n}}(y)$. Thus, $z \in C$. □

It now follows that if X is SEULAC, then X is ELC.

It only remains to show that every EULAC space is SEULAC. To do so, we will need the tools in the next section.

6. Witnessing chains and arc chains

In this section, we assume X is ELC. Fix a function $f : \mathbb{N} \to \mathbb{N}$ that witnesses that X is ELC.

Definition 6.1. A *witnessing chain* is a sequence $(m, w, w_1, \ldots, w_k)$ such that

- $I^n(w_i) \cap I^n(w_{i+1}) \cap X \neq \emptyset$,
- $B_{2^{-m}}(I^n(w_i)) \subseteq I^n(w)$, and
- $\mathrm{diam}(I^n(w_i)) < 2^{-f(m)}$.

We should note that a witnessing chain is not a chain of sets but a formal representation of a chain of sets. We now make some notation. Let $\omega = (m, w, w_1, \ldots, w_k)$ be a witnessing chain. We let:

$$
\begin{aligned}
V_\omega &= \bigcup_{i=1}^{k} B_{2^{-m}}(I^n(w_i)) \\
m_\omega &= m \\
k_\omega &= k \\
w_{\omega,j} &= w_j \\
w_\omega &= w
\end{aligned}
$$

Proposition 6.2. *Suppose $\omega = (m, w, w_1, \ldots, w_k)$ is a witnessing chain and that $1 \leq j \leq k$. Then, for all $x, y \in I^n(w_j) \cap X$,*

$$C_x(B_{2^{-m}}(I^n(w_i))) = C_y(B_{2^{-m}}(I^n(w_i))).$$

Proof. Let $U = B_{2^{-m}}(I_n(w_i))$. Since $\mathrm{diam}(I^n(w_i)) < 2^{-f(m)}$, it follows that $y \in X \cap B_{2^{-f(m)}}(x)$. Since $B_{2^{-f(m)}}(x) \subseteq U$, it follows that

$$C_x(B_{2^{-m}}(x)) \subseteq C_x(U).$$

Since $X \cap B_{2^{-f(m)}}(x) \subseteq C_x(B_{2^{-m}}(x))$, it follows that $y \in C_x(U)$. Hence, $C_x(U) = C_y(U)$. □

We make some more notation regarding witnessing chains. Suppose ω is a witnessing chain and that $1 \leq i \leq k_\omega$. We let

$$C_{\omega,i} = C_x(B_{2^{-m_\omega}}(I^n(w_{\omega,i}))$$

for any $x \in I^n(w_i) \cap X$. Hence, $C_{\omega,i}$ is a connected open subset of X. In addition, $C_{\omega,i} \cap C_{\omega,i+1} \neq \emptyset$ if $i < k_\omega$. It follows that

$$C_\omega =_{df} \bigcup_{j=1}^{k} C_{\omega,j}$$

is an open connected subset of X, and

$$C_\omega \subseteq V_\omega.$$

Lemma 6.3. *For every $x, y \in I^n(w) \cap X$, x and y are in the same connected component of $I^n(w) \cap X$ if and only if there is a witnessing chain $(m, w, w_1, \ldots, w_k)$ with $x \in I^n(w_1)$ and $y \in I^n(w_k)$. Furthermore, if x, y are in the same connected component of $I^n(w) \cap X$, then for every $\epsilon > 0$, there is such a witnessing chain ω for which it is also true that $-m_\omega < \ln(\epsilon)$.*

Proof. Suppose x and y are in the same connected component of $I^n(w) \cap X$. Let A be an arc from x to y such that $A \subseteq I^n(w) \cap X$. Choose m so that $d(A, \mathbb{R}^n - I^n(w)) \geq 2^{-(m-1)}$ and $2^{-m} < \epsilon$. For all $p \in A$, choose u so that $p \in I^n(u)$, $I^n(u) \subseteq B_{2^{-m}}(p) \subseteq I^n(w)$, and $\mathrm{diam}(I^n(u)) < 2^{-f(m)}$. From these rectangles, we can choose $I^n(u_1), \ldots, I^n(u_k)$ that cover A and such that $(I^n(u_1) \cap A, \ldots, I^n(u_k) \cap A)$ is a simple chain. Since $d(A, \mathbb{R}^n - I^n(w)) \geq 2^{-(m-1)}$, and since $I^n(u_i) \subseteq B_{2^{-m}}(p)$ for some $p \in A$, it follows that $B_{2^{-m}}(I^n(u_i)) \subseteq I^n(w)$. It follows that $(m, w, u_1, \ldots, u_k)$ is a witnessing chain.

Conversely, suppose we have a witnessing chain $\omega = (m, w, w_1, \ldots, w_k)$ such that $x \in I^n(w_1)$ and $y \in I^n(w_k)$. Then, $x, y \in C_\omega$. Thus, x, y are in the same connected component of $I^n(w) \cap X$. □

Definition 6.4. Suppose $\omega = (m, w, w_1, \ldots, w_k)$ is a witnessing chain. If $x \in I^n(w_1) \cap X$, and if $y \in I^n(w_k) \cap X$, then we say that ω *witnesses* that x, y are in the same connected component of $I^n(w) \cap X$.

We note that the set of witnessing chains is computably enumerable.

Definition 6.5. An *arc chain* is a sequence of the form $(\omega_1, \ldots, \omega_l)$ where

- ω_j is a witnessing chain,
- $(V_{\omega_1}, \ldots, V_{\omega_l})$ is a simple chain, and
- $X \cap I^n(w_{\omega_i, k_{\omega_i}}) \cap I^n(w_{\omega_{i+1}, 1}) \neq \emptyset$.

We note that the set of arc chains is computably enumerable.

We make some notation. Let $\mathfrak{p} = (\omega_1, \ldots, \omega_l)$ be an arc chain. We let:

$$\begin{aligned} l_{\mathfrak{p}} &= l \\ \omega_{\mathfrak{p},j} &= \omega_j \\ V_{\mathfrak{p},j} &= V_{\omega_{\mathfrak{p},j}} \\ C_{\mathfrak{p},j,i} &= C_{\omega_j,i} \\ C_{\mathfrak{p},j} &= C_{\omega_j} \\ C_{\mathfrak{p}} &= \bigcup_j C_{\mathfrak{p},j} \end{aligned}$$

Definition 6.6. Suppose $\mathfrak{p}_0$ and $\mathfrak{p}_1$ are arc chains. We say that $\mathfrak{p}_0$ *refines* $\mathfrak{p}_1$ if $(V_{\mathfrak{p}_0,1}, \ldots, V_{\mathfrak{p}_0,l_{\mathfrak{p}_0}})$ refines $(V_{\mathfrak{p}_1,1}, \ldots, V_{\mathfrak{p}_1,l_{\mathfrak{p}_1}})$. We say that $\mathfrak{p}_0$ *goes straight through* $\mathfrak{p}_1$ if $(V_{\mathfrak{p}_0,1}, \ldots, V_{\mathfrak{p}_0,l_{\mathfrak{p}_0}})$ goes straight through $(V_{\mathfrak{p}_1,1}, \ldots, V_{\mathfrak{p}_1,l_{\mathfrak{p}_1}})$.

Definition 6.7. If $\mathfrak{p}$ is an arc chain, then the *diameter* of $\mathfrak{p}$ is defined to be the diameter of the simple chain $(V_{\mathfrak{p},1}, \ldots, V_{\mathfrak{p},l_\mathfrak{p}})$.

Definition 6.8. If $\mathfrak{p}$ is an arc chain such that $(V_{\mathfrak{p},1}, \ldots, V_{\mathfrak{p},l_\mathfrak{p}})$ is a simple chain from x to y, then we say that $\mathfrak{p}$ is an *arc chain from x to y*.

Theorem 6.9. *Suppose $\mathfrak{p}$ is an arc chain from x to y. Then, for every $\epsilon > 0$, there is a refinement of $\mathfrak{p}$ of diameter less than ϵ that is also an arc chain from x to y and that goes through $\mathfrak{p}$.*

Proof. Let $\omega_i = \omega_{\mathfrak{p},i}$. Let

$$\begin{aligned} w_i &= w_{\omega_i} \\ m_i &= m_{\omega_i} \\ k_i &= k_{\omega_i} \\ w_{i,j} &= w_{\omega_i,j} \\ l &= l_\mathfrak{p} \end{aligned}$$

Let $x_0 = x$, $x_l = y$, and let

$$x_i \in I^n(w_{i,k_i}) \cap I^n(w_{i+1,1}) \cap X$$

for $0 < i < l$. Hence, x_{i-1}, x_i are in the same connected component of $V_{\omega_i} \cap X$ for $i = 1, \ldots, l$.

Let B_1 be an arc in C_{ω_1} from x_0 to x_1. There exists $q_1 \in B_1$ such that every point on B_1 between q_1 and x_1 is in $I^n(w_{1,k_1}) \cap I^n(w_{2,1}) \cap X$. There exist $s_1 \in \mathbb{N}$ and $x_{1,0}, \ldots, x_{1,s_1} \in B_1$ with the following properties.

- $x_{1,0} = x_0$.
- $x_{1,s_1} = x_1$.
- $x_{1,i}$ is between $x_{1,i-1}$ and $x_{1,i+1}$ on B_1,
- q_1 is between x_{1,s_1-1} and x_{1,s_1} on B_1.
- $d(x_{1,i}, x_{1,i+1}) < \epsilon/3$.

Let $B_{1,j}$ be the arc on B_1 from $x_{1,j-1}$ to $x_{1,j}$ for $j = 1, \ldots, s_1$.

Now, let D be the result of removing the arc on B_1 from x_0 to q_1 from C_{ω_2}. Hence, D is open in X. We claim that there is an arc in D from x_1 to x_2. For, let g witness that X is EULAC. For each $p \in D$, let $\epsilon_p > 0$ be such that $B_{\epsilon_p}(p) \cap X \subseteq D$. Let m_p be such that $-m_p < \ln(\epsilon_p)$. Let $U_p = B_{2^{-g(m_p)}}(p) \cap X$. Then, there is a simple chain $U_{p_1}, \ldots, U_{p_k}$ such that $x_1 \in U_{p_1}$ and $x_2 \in U_{p_k}$. It follows that there is a path in D from x_1 to x_2. It then follows that there is an arc in D from x_1 to x_2. Call this arc B_2. Note that the intersection of B_2 and B_1 is contained in the arc on B_1 from q_1 to x_1.

Choose q_2 so that every point on B_2 between q_2 and x_2 is in $I^n(w_{2,k_2}) \cap I^n(w_{3,1}) \cap X$. There exists $s_2 \in \mathbb{N}$ and $x_{2,0}, \ldots, x_{2,s_2} \in B_2$ such that

- $x_{2,0} = x_1$,
- $x_{2,s_2} = x_2$,
- $x_{2,i}$ is between $x_{2,i-1}$ and $x_{2,i+1}$ on B_2,
- q_2 is between x_{2,s_2-1} and x_{2,s_2} on B_2, and
- $d(x_{2,i}, x_{2,i+1}) < \epsilon/3$.

We repeat this procedure with $V_{\omega_3}, \ldots, V_{\omega_l}$. We obtain arcs

$$B_3, \ldots, B_l, B_{3,1}, \ldots, B_{l,s_l},$$

and points

$$x_{3,0}, \ldots, x_{l,s_l}, q_3, \ldots, q_{l-1}.$$

The distance between any two non-adjacent arcs in $\{B_{1,1}, \ldots, B_{l,s_l}\}$ is positive. Let δ be a positive lower bound on these distances.

We form a witnessing chain $\omega_{i,t}$ as follows. First of all, choose m such that $2^{-m} < \min\{\delta/8, \epsilon/3\}$. We then choose $w_1, \ldots, w_k$ with the following properties.

- $B_{2^{-m}}(I^n(w_j)) \subseteq V_{\omega_i}$,
- $I^n(w_j) \cap I^n(w_{j+1}) \cap B_{i,t} \neq \emptyset$,
- $\text{diam}(I^n(w_j)) < 2^{-f(m)}$,
- $B_{i,t} \subseteq \bigcup_j I^n(w_j)$, and
- $x_{i,t-1} \in I^n(w_1) \cap X$, and $x_{i,t} \in I^n(w_k) \cap X$.

Choose w such that $B_{2^{-m}}(I^n(w_j)) \subseteq I^n(w)$ for all j. Let $\omega_{i,s} = (m, w, w_1, \ldots, w_k)$.

It follows that $(V_{\omega_{1,1}}, \ldots, V_{\omega_{l,s_l}})$ is a simple chain that refines and goes straight through $(V_{\omega_1}, \ldots, V_{\omega_l})$. In addition, its diameter is less than ϵ. In addition,

$$x_{i,j} \in I^n(w_{\omega_{i,j}, k_{\omega_{i,j}}}) \cap I^n(w_{\omega_{i,j+1},1}) \cap X,$$

and

$$x_{1,s_1} \in I^n(w_{\omega_{1,s_1}, k_{\omega_{1,s_1}}}) \cap I^n(w_{\omega_{i+1,1},1}) \cap X.$$

It follows that $(\omega_{1,1}, \ldots, \omega_{l,s_l})$ refines and goes straight through $(\omega_1, \ldots, \omega_l)$. □

We are now in position to complete the proof of our main result.

7. Every EULAC space is SEULAC

Theorem 7.1. *If X is EULAC, then X is SEULAC.*

Proof. Suppose X is ELC. Let $g : \mathbb{N} \to \mathbb{N}$ witness that X is EULAC. We assume g is increasing. Let $f(m) = g(m+1)$. Therefore, as in the proof of Theorem 5.2, f witnesses that X is EULAC, and f witnesses that X is ELC.

Now, let $w \in \Sigma^*$ and $p, r \in \Sigma^\omega$ be given as input. Assume $w \in dom(\nu_{\mathbb{N}})$, and let $m = \nu_{\mathbb{N}}(w)$. Read p, r while cycling through Σ^* until u, v, w' are found such that

$$\begin{aligned}
\iota(u) &\lhd p \\
\iota(v) &\lhd r \\
I^n(u) \cap X &\neq \emptyset \\
I^n(v) \cap X &\neq \emptyset \\
I^n(u), I^n(v) &\subseteq I^n(w') \\
\text{diam}(I^n(w')) &< 2^{-m} \\
d(\overline{I^n(u)}, \overline{I^n(v)}) &< 2^{-f(m)} \\
d(\overline{I^n(u)}, \mathbb{R}^n - I^n(w')) &\geq 2^{-(m+1)} \\
d(\overline{I^n(v)}, \mathbb{R}^n - I^n(w')) &\geq 2^{-(m+1)}
\end{aligned}$$

Let $x = \rho^n(p)$, and let $y = \rho^n(r)$. Hence, there is an arc from x to y of diameter less than $2^{-(m+1)}$. Also, any such arc is contained in $I^n(w')$. In addition, the diameter of any arc from x to y contained in $I^n(w')$ is less than 2^{-m}.

From p, r, we can compute a sequence of arc chains $\mathfrak{p}_0, \mathfrak{p}_1, \ldots$ with the following properties. For the sake of stating these properties more compactly, let $V_{i,j} = V_{\mathfrak{p}_i,j}$ and $l_i = l_{\mathfrak{p}_i}$.

(1) $\bigcup_j V_{i,j} \subseteq I^n(w')$,
(2) $\mathfrak{p}_{i+1}$ refines and goes straight through $\mathfrak{p}_i$.
(3) The diameter of $\mathfrak{p}_i$ is less than 2^{-i}.
(4) There exists u such that $\iota(u) \lhd p$ and

$$\overline{I^n(u)} \subseteq V_{i,1} - \bigcup_{2 \leq j \leq l_i} V_{i,j}.$$

(5) There exists v such that $\iota(v) \lhd r$ and

$$\overline{I^n(v)} \subseteq V_{i,l_i} - \bigcup_{2 \leq j \leq l_i} V_{i,j}.$$

Let

$$\begin{aligned} s(i+1, 0) &= 0 \\ s(i+1, t+1) &= \max\{j \mid \forall j' \in (s(i+1,t), j]\ V_{i+1,j'} \subseteq V_{i,t+1}\}. \end{aligned}$$

As we compute $\mathfrak{p}_0, \mathfrak{p}_1, \ldots$, we simulataneously compute an array of rational intervals $\{I_{i,j}\}_{i,j=1,\ldots,l_i}$ such that the following hold.

- $\{I_{i,j}\}_{j=1,\ldots,l_i}$ is a simple chain that covers $[0,1]$.
- $\{I_{i+1,j}\}_j$ refines $\{I_{i,j}\}_j$.
- $\lim_{i\to\infty} \max_j \operatorname{diam}(\overline{I_{i,j}}) = 0$.
- $I_{i+1,j'} \subseteq I_{i,j}$ if and only if $s(i+1, j-1) < j' \leq s(i+1, j)$.

Let $r' \in \Sigma^\omega$ suitably encode a list of all pairs (I, J) such that $R_{i,j} \subseteq J$ and $\overline{I} \subseteq I_{i,j}$ for some i, j.

We claim that r' is a name of a function f. For, let $x \in [0,1]$. We claim there exists $z \in X$ such that $z \in I^n(v)$ for all v such that $(I, I^n(v))$ is listed by r' for some I. By way of contradiction, suppose otherwise. It follows that there exist $(i_1, j_1), \ldots, (i_t, j_t)$ such that $x \in I_{i_1,j_1} \cap \ldots \cap I_{i_t,j_t}$ but $V_{i_1,j_1} \cap \ldots \cap V_{i_t,j_t} = \emptyset$. It follows that $I_{i_1,j_1}, \ldots, I_{i_t,j_t}$ are not linearly ordered by inclusion (otherwise, $V_{i_1,j_1}, \ldots, V_{i_t,j_t}$ would be). Suppose $I_{a,b}$ and $I_{c,d}$ are two $\subseteq$-minimal intervals in this set. Without loss of generality, suppose $a \leq c$. There exist unique b' such that $I_{c,d} \subseteq I_{a,b'}$. It follows that $|b - b'| = 1$. Since $\{I_{i,j}\}_j$ is simple, it now follows that there are at most two such $\subseteq$-minimal intervals. Suppose $I_{a,b}$ and $I_{c,d}$ *are* these intervals. There exists d' such that $|d - d'| = 1$ and $I_{c,d'} \subseteq I_{a,b}$. Choose $z \in V_{c,d'} \cap V_{c,d} \cap X$. It now follows that $z \in V_{i_1,j_1} \cap \ldots \cap V_{i_t,j_t} \cap X$- a contradiction.

Hence, there exists $z \in X$ such that $z \in I^n(v)$ for all v such that $(I, I^n(v))$ is listed by r' for some I. Since $\operatorname{diam}(V_{i,j}) < 2^{-i}$, it now follows that z is unique. Hence r' is a name of a function f.

We now claim f is one-to-one. To this end, we first note that $f[I_{i,j}] \subseteq V_{i,j}$. Let $0 \leq x_1 < x_2 \leq 1$. There exist i, j_1, j_2 such that $x_1 \in I_{i,j_1}$, $x_2 \in I_{i,j_2}$, and $|j_1 - j_2| > 1$. Hence, $V_{i,j_1} \cap V_{i,j_2} = \emptyset$.

Since $x \in V_{i,1}$ and $y \in V_{i,l_i}$, it follows that $f(0) = x$ and $f(1) = y$. This completes the proof. □

The above proof not only constructs an arc from x to y, but in addition it constructs a parameterization of that arc. We note that J. Miller has constructed an arc that is computable as a compact subset of $\mathbb{R}^2$ and has computable endpoints but has no computable parameterization [13]. Recently, Gu, Lutz, and Mayordomo, have strengthened this result by constructing an arc A such that any computable function f of $[0,1]$ onto A retraces itself infinitely often [7].

The following has already been claimed by J. Miller in [13].

Corollary 7.2. *An arc $A \subseteq \mathbb{R}^n$ has a computable parameterization if and only if it is computable as a compact set and is effectively locally connected.*

8. Prospects for uniformity

Thus far, we have concentrated on non-uniform results. That is, we have merely proven that when certain given data are computable then some other associated data are necessarily computable as well. In contrast, a uniform result shows that the associated data can be computed by a single Turing machine from the given data even when the given data are not computable. Such results should always be sought since they have the widest range of applicability.

Our results are all based on Lemma 4.2, the Computable Lebesgue Number Lemma. As noted before, our proof of this result is not uniform. However, it is easily checked that it can be made uniform if we exclude single-point spaces. This is a sufficiently wide range of application. *Hence, throughout the rest of this section, X is a non-degenerate continuum in $\mathbb{R}^n$.*

When passing from non-uniform to uniform results, one must sometimes revise the underlying effective notions so as to merely isolate the essential data without imposing the restriction of computability. This leads to the following.

Definition 8.1. A *local connectivity witness* for X is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$ and all $x \in X$, $X \cap B_{2^{-f(k)}}(x) \subseteq C_x(B_{2^{-k}}(x))$.

Definition 8.2. A *uniformly local arcwise connectivity witness* for X is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$ and all $x, y \in X$, if $d(x,y) < 2^{-f(k)}$, then x, are joined by an arc in X of diameter less than 2^{-k}.

Definition 8.3. A pair (f, g) is called a *strong witness* of uniformly local arcwise connectivity for X if f witnesses the uniformly local arcwise connectivity of X and if $g :\subseteq \mathbb{N} \times \Sigma^\omega \times \Sigma^\omega \times \Sigma^\omega \to \Sigma^\omega$ is such that whenever $k \in N$ and p, q are names of points $x, y \in X$ respectively such that $d(x,y) < 2^{-f(k)}$, $g(k,p,q)$ is a name of a parameterization of an arc in X from x to y.

We now state the uniform versions of our results. The proofs are simple and standard modifications of those of the corresponding non-uniform results.

Lemma 8.4 (Uniformly Computable Lebesgue Number Theorem). *From a natural number n, a name of a non-degenerate continuum $X \subseteq \mathbb{R}^n$, and a $w \in \Sigma^*$ such that*

$$\{I^n(u) \mid \iota(u) \lhd w\}$$

is a covering of X, it is possible to compute a natural number k with the property that 2^{-k} is a Lebesgue Number for this covering of X.

Theorem 8.5. *Each of the following can be uniformly computed from the other.*

(1) *A natural number* n*, a name of a non-degenerate continuum* $X \subseteq \mathbb{R}^n$*, and a local connectivity witness for* X*.*
(2) *A natural number* n*, a name of a non-degenerate continuum* $X \subseteq \mathbb{R}^n$*, and a strong witness of uniformly local arcwise connectivity for* X*.*

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