COMPUTING CONFORMAL MAPS ONTO CIRCULAR DOMAINS

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Dedicated to Professor Klaus Weihrauch on the occasion of his 65th birthday and to the memory of Marian Pour-El.

Abstract. We show that, given a non-degenerate, finitely connected domain D, its boundary, and the number of its boundary components, it is possible to compute a conformal mapping of D onto a circular domain *without* prior knowledge of the circular domain. We do so by computing a suitable bound on the error in the Koebe construction (but, again, without knowing the circular domain in advance). Recent results on the distortion of capacity by Thurman [25] and the computation of capacity by Ransford and Rostand [24] are used. As a scientifically sound model of computation with continuous data, we use an informal version of Type-Two Effectivity as developed by Kreitz and Weihrauch [27].

1. Introduction

Let $\hat{\mathbb{C}}$ denote the extended complex plane, $\mathbb{C} \cup \{\infty\}$. By a *domain*, we mean an open connected subset of $\hat{\mathbb{C}}$. A domain is *n-connected* if its complement has exactly n connected components. We will call a domain *degenerate* if a component of its complement consists of a single point.

We begin by discussing the history of the problem we are to consider. The Riemann Mapping Theorem states that every non-degenerate 1-connected domain is conformally equivalent to the unit disk, $\mathbb{D}$. Hence, there is a single *canonical domain* to which all non-degenerate 1-connected domains are conformally equivalent. With regards to the situation for 2-connected domains and higher, we must first note that it is not the case that all n-connected domains are conformally equivalent when $n \geq 2$. In fact, two annuli are conformally equivalent only when the ratio of their inner to outer radii are the same. Hence, much research has focused on proving existence of conformal mappings onto various kinds of canonical domains. See [21] and [12]. Some existence proofs rely on extremal-value arguments and hence are not constructive. At the same time, there are explicit formulae for the conformal mappings from a multiply connected domain to a circular slit disk and a circular slit annulus. Unlike the simply connected case, every multiply connected domain can be identified by $3n-6$ parameters, where n is the connectivity of the domain. But even if we know one type canonical domain to which a given domain is conformally equivalent, *we are still not wiser about the configuration of other canonical domains to which the given domain is conformally equivalent.*

Among the canonical domains are the *circular domains* which are obtained by deleting one or more disjoint closed disks from the plane. These domains are the

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canonical domains in a number of recent studies of the Schwarz-Christoffel formula for multiply connected domains, nonlinear problems in mechanics, and in aircraft engineering. See, for example, [4] and [6]. Jeong and Mityushev have recently obtained explicit formulae for the Green's function of a circular domain [17].

In 1910, Paul Koebe put forth and investigated an iterative technique for approximating the unique conformal map f_D of a non-degenerate n-connected domain D with $\infty \in D$ and $0 \notin D$ onto a circular domain C_D of the form $z + O(z^{-1})$ [18]. This method is now known as the *Koebe Construction* and will be described in Section 2. In 1959, Dieter Gaier calculated upper bounds on the error in this construction which tend to zero as the iterations progress [8]. However, these bounds use certain numbers associated with C_D. Hence, to apply Gaier's bounds for the sake of approximating f_D, one must first know a fair amount about the circular domain C_D. In [15], Henrici presents a modification of Gaier's construction. But, again, Henrici's bounds use certain numbers associated with the circular domain C_D which usually is not known in advance.

It is a pleasing coincidence that the problems of computing a conformal map of an n-connected domain with $n > 1$ onto various canonical domains were considered not only by such outstanding figures in complex analysis as we have mentioned here, but also appeared as questions in computable analysis in the pioneering book by Pour-El and Richards [22].

In this paper, using a fairly recent result by Thurman on the distortion of capacity by conformal maps [25] and a very recent result by Ransford and Rostand on the computation of capacity [24], we will overcome these difficulties by showing how to compute sufficiently good estimates of the numbers in Henrici's bound from the domain D and its boundary and without any prior knowledge of any information about the circular domain C_D.[1] We will *not* give explicit formulas for our approximations to these quantities. Rather, we will show how to obtain sufficiently good approximations to them from sufficiently good approximations to D and its boundary.

How we approximate an open set and its boundary requires some explanation. In addition, we want to use a scientifically sound model of computation with continuous data to prove the correctness of our results. The mathematical theory of computation with discrete data is *computability theory*, and the mathematical theory of computation with continuous data is *computable analysis*. To address these issues, we will use the Type-Two Effectivity approach to computable analysis, which is with full rigor and precision described in Weihrauch's sterling book [27]. However, we will try to sidestep much of the notation in this approach and use a somewhat informal presentation which is suitable for these applications. We use Type-Two Effectivity because there is a considerable body of very useful foundational work based on this approach. It also facilely deals with computations on hyperspaces such as the set of all open subsets of the plane as well as more concrete spaces such as the real line. We will also make copious use of the results in Hertling's paper on the Effective Riemann Mapping Theorem [16]. It should be emphasized though that we are not merely translating known existence proofs into

[1] It is our understanding that R. Rettinger has shown how to compute the conformal mapping onto the canonical domain obtained by cross-cutting an annulus with $n - 2$ concentric arcs and has obtained some complexity results as well.

the format of Type-Two Effectivity, nor are we reproving known existence results in a constructive manner.

The outline of our work is as follows. In Section 2 we describe the Koebe Construction. In Section 3, we then develop the basics of computable analysis and Type Two Effectivity in as informal a manner as we can. We then prove some preliminary results related to computation on $\hat{\mathbb{C}}$ and its hyperspaces. In Section 4, we use these results to show that the sequences generated by the Koebe Construction can be computed from the data $D, \partial D$ and the number of boundary components. As with many problems in conformal mapping, harmonic measure will play a key role in our solution. So, in Section 5, we rigorously develop the basics of computing (in the format of Type Two Effectivity) with harmonic functions including differentiation, harmonic extension, and solutions to Dirichlet problems on finitely connected Jordan domains. The latter will also lead us to prove a computable version of Carathéodory's Theorem. In section 6, we will demonstrate the computability of harmonic measure, the Riemann matrix, and capacity of a finitely connected Jordan domain. With these preliminaries out of the way, we will in Section 7 show how to compute suitable estimates on the numbers used in Henrici's error bound from sufficiently good estimates to D and its boundary. In Section 8, we prove our main results. Finally, in Section 9, we investigate the necessity of the parameter ∂D. It is well-known that the boundary of a domain can not be computed from the domain itself (in the sense we describe later). It then follows from the results of Hertling [16] that f_D can not be computed from D alone. However, we show that arbitrarily good approximations to ∂D can be computed from sufficiently good approximations to D, f_D, and ∂C_D. So, we may conclude that the boundary of D represents the least amount of additional information necessary to compute f_D from D.

We have tried to write this paper so that it will be accessible to researchers in computer science, complex analysis, and logic. We have therefore included many fundamental definitions and results from these fields. In some cases, we have merely stated an informal outline of the pertinent facts. We have cited references where the reader may obtain more complete explanations.

2. The Koebe Construction

Figures 1 through 5 illustrate the construction when $n = 3$. Let D be a non-degenerate n-connected domain that contains ∞ but not 0. We inductively define sequences $\{D_k\}_{k=0}^{\infty}$, $\{D_{k,1}\}_{k=0}^{\infty}$, ..., $\{D_{k,n}\}_{k=0}^{\infty}$, and $\{f_k\}_{k=0}^{\infty}$ as follows.

To begin, let $D_{0,1}, \ldots, D_{0,n}$ be the connected components of $\hat{\mathbb{C}} - D$. Let $D_0 = D$. Let $f_0 = Id_D$.

Let $k \in \mathbb{N}$, and suppose $f_k, D_k, D_{k,1}, \ldots, D_{k,n}$ have been defined. Let $k' \in \{1, \ldots, n\}$ be equivalent to $k + 1$ modulo n. Let f_{k+1} be the conformal map of $\hat{\mathbb{C}} - D_{k,k'}$ onto a circular domain C such that $f_{k+1}(z) = z + O(z^{-1})$. Now, let $D_{k+1} = f_{k+1}[D_k]$. Let $D_{k+1,j} = f_{k+1}[D_{k,j}]$ when $j \neq k'$. Let $D_{k+1,k'} = \hat{\mathbb{C}} - C$.

Let $g_k = f_k \circ \ldots \circ f_0$. It is well-known that $\lim_{k\to\infty} g_k = f_D$.

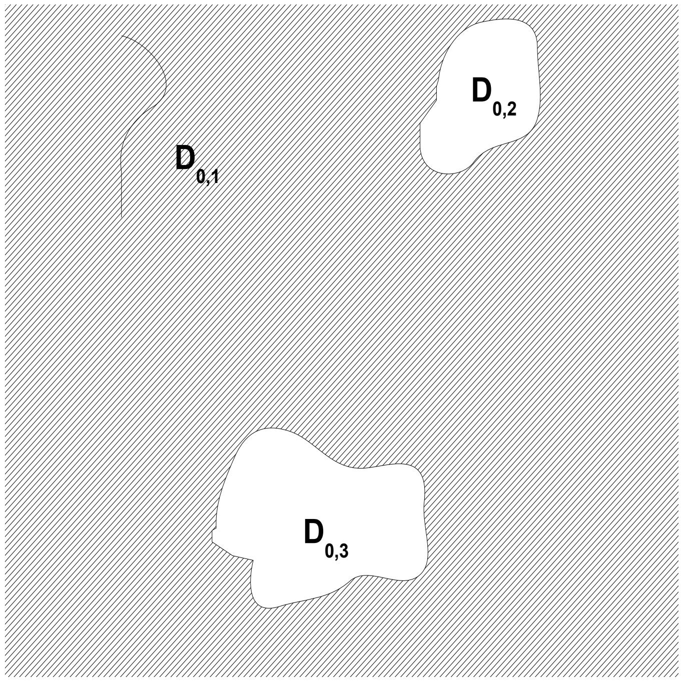


FIGURE 1. The Koebe Construction, $n = 3$, initial state

In order to state Henrici's bound on the error in this construction, we now introduce some numbers associated with a circular domain. Suppose C is a circular domain and that $\Gamma_1, \ldots, \Gamma_n$ are the components of $\hat{\mathbb{C}} - C$. Whenever, $j, k \in \{1, \ldots, n\}$ are distinct, let Γ_j^k be the disk obtained by reflecting Γ_k into Γ_j. Then,

$$\delta_C =_{df} \min\{d(\Gamma_j, \Gamma_j^k) \mid j, k \in \{1, \ldots, n\} \ \wedge \ j \neq k\}.$$

(See page 502 of [15].) Let $D_R(z)$ denote the disk of radius R and center z. We also let

$$\rho_C = \min\{R > 0 \mid \hat{\mathbb{C}} - C \subseteq D_R(0)\}.$$

Suppose $\Gamma_j = \overline{D_{r_j}(z_j)}$. We define μ_C to be the reciprocal of

$$\min\{r > 0 \mid \exists j, k \in \{1, \ldots, n\} \ (j \neq k \ \wedge \ \overline{D_{rr_j}(z_j)} \cap \overline{D_{rr_k}(z_k)} \neq \emptyset)\}.$$

Note that $\mu_C^{-1} > 1$. Finally, let

$$\gamma_C =_{df} \frac{2\rho_C^2}{\pi \delta_C} \left[\frac{2[\pi \mu_C^{-1}]^2}{\ln \mu_C^{-1}} + 1 \right].$$

(The constants ρ_C, μ_C, γ_C are defined on page 505 of [15].) We now define these quantities for D by letting

$$\begin{aligned} \delta_D &= \delta_{C_D} \\ \mu_D &= \mu_{C_D} \\ \rho_D &= \rho_{C_D} \\ \gamma_D &= \gamma_{C_D} \end{aligned}$$

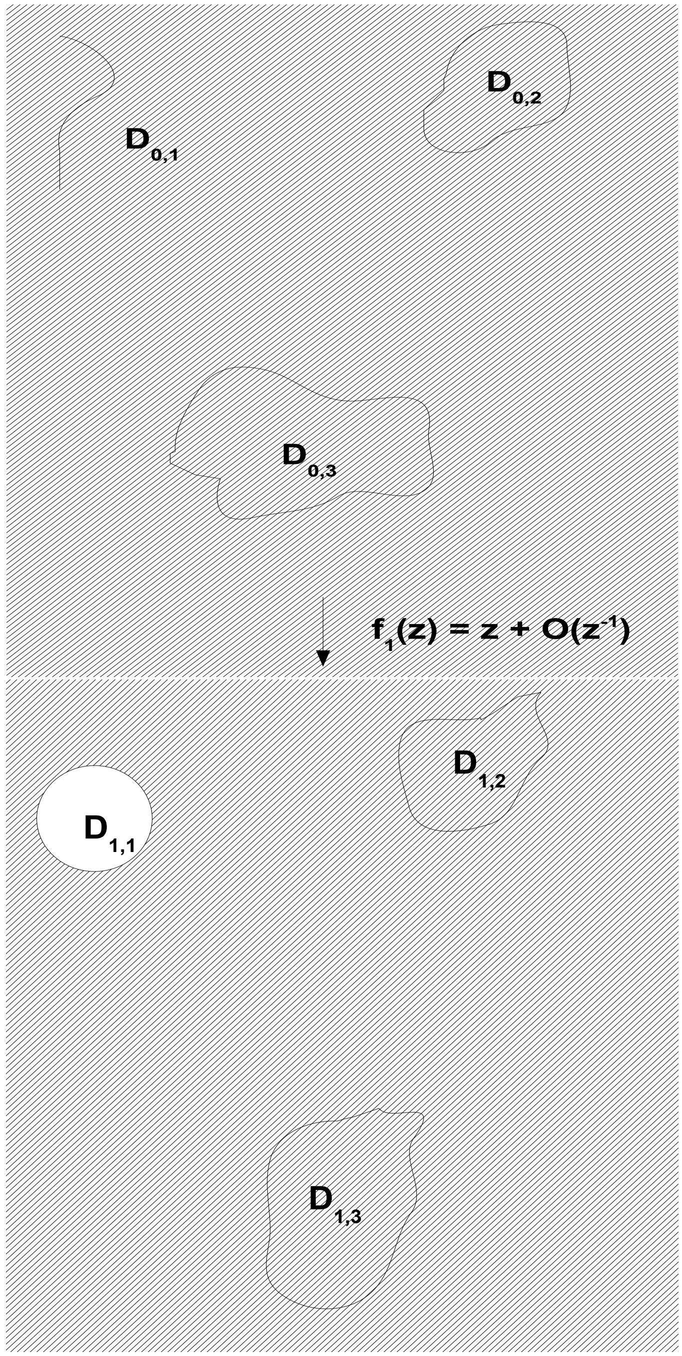
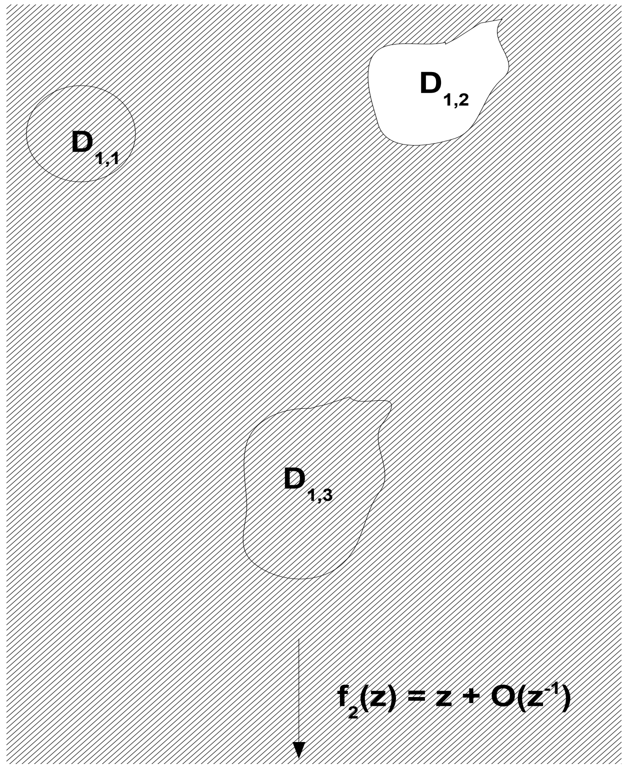


FIGURE 2. The Koebe Construction, $n = 3$, first iteration

The following is Theorem 17.7A of [15].

Theorem 2.1. *For all* $z \in D - \{\infty\}$ *and all* $j \in \mathbb{N}$, $|g_j(z) - f_D(z)| \leq \gamma_D \mu_D^{4\lfloor j/n \rfloor}$.



3. Preliminaries from computable analysis

We will rely on an intuitive understanding of terms such as 'algorithm' and 'procedure'. The formal mathematical formulation of these concepts is a *Turing machine*, and for this we refer the reader to standard references such as [5].

Type-Two Effectivity is based on *naming systems* for uncountable spaces. These come in many varieties, but the sort we want to consider, and the kind suitable for computation, are those in which a name for an object is, informally, a list of approximations to that object. Computations on these spaces are viewed as transforming names to names. We now begin to specify all this for the situations we wish to consider.

3.1. **Naming systems for the extended complex plane and associated hyperspaces.** In the following, a list is just a countably infinite sequence of objects, possibly with repetitions, indexed by $\mathbb{N} = \{0, 1, 2, \ldots\}$.

We first define a subbasis for the standard topology on $\hat{\mathbb{C}}$ as follows. Call an open rectangle R *rational* if the coördinates of its vertices are all rational numbers. We first declare all rational rectangles to be subbasic. We then declare to be subbasic all sets of the form $\hat{\mathbb{C}} - \overline{R}$ where R is a rational rectangle that contains 0. A *name* of a point $z \in \hat{\mathbb{C}}$ is a list of all the subbasic neighborhoods that contain z. Hence, every point in $\hat{\mathbb{C}}$ has many names (there are infinitely many ways to list all the neighborhoods that contain z). However, no two points can have a common name.

Let $\mathcal{O}(\hat{\mathbb{C}})$ be the set of all open subsets of $\hat{\mathbb{C}}$. We now define a name for an open $U \subseteq \hat{\mathbb{C}}$ to be a list of all subbasic open sets whose closures are contained in U. Let

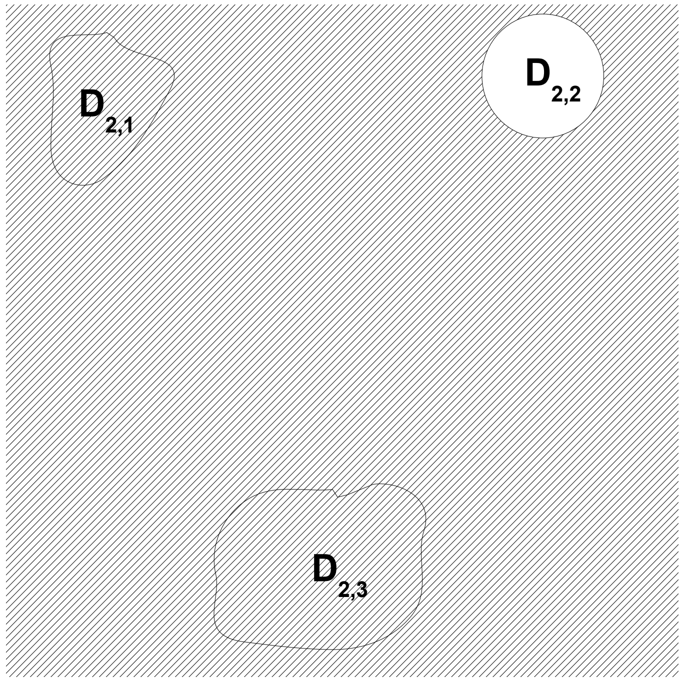
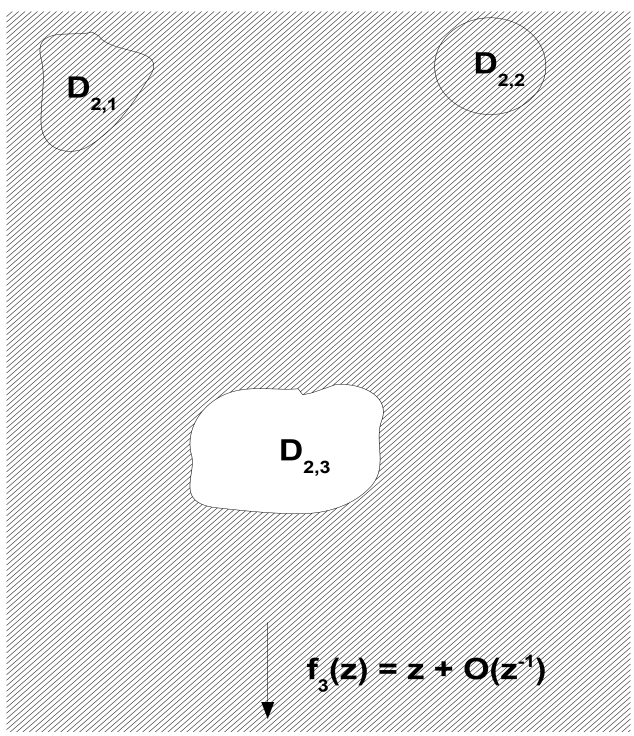


FIGURE 3. The Koebe Construction, $n = 3$, second iteration



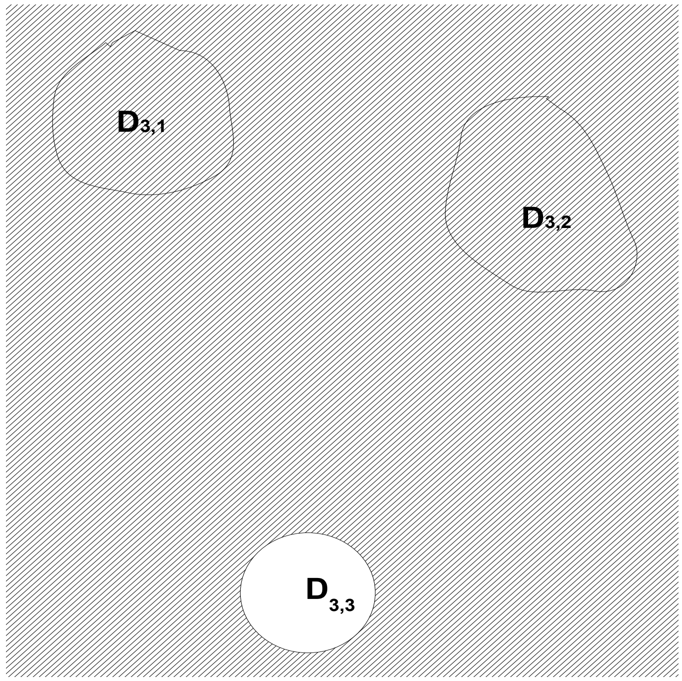


FIGURE 4. The Koebe Construction, $n = 3$, third iteration

$\mathcal{C}(\hat{\mathbb{C}})$ be the set of all closed subsets of $\hat{\mathbb{C}}$. A name of a closed $C \subseteq \hat{\mathbb{C}}$ is a list of all subbasic open sets that intersect C. In each of these cases, a set will have many names, but no two sets can have a common name.

Let $f :\subseteq A \to B$ denote that f is a function whose domain is contained in A and whose range is contained in B. Let $C_{open}(\subseteq \hat{\mathbb{C}})$ be the set of all continuous functions $f :\subseteq \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\text{dom}(f)$ is open. A name of such a function f is a list of all pairs of the form (U, V) such that U, V are subbasic open sets, $\overline{U} \subseteq \text{dom}(f)$, and $f[U] \subseteq V$. Let $C_{closed}(\subseteq \hat{\mathbb{C}})$ denote the set of all continuous functions $f :\subseteq \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\text{dom}(f)$ is closed. A name of such a function f is a list of pairs of subbasic open sets $\{(U_n, V_n)\}_{n\in\mathbb{N}}$ that possesses the following two properties.

(1) Each U_n contains at least one point of the domain of f and its intersection with the domain of f is mapped into V_n by f.
(2) Whenever x is in the domain of f and U, V are subbasic open sets that that U contains x and V contains $f(x)$, there exists n such that $x \in U_n \subseteq U$ and $V_n \subseteq V$.

The first property guarantees that each (U_n, V_n) provides accurate partial information about f. The seecond ensures that we can obtain arbitrarily good estimates to any value of $f(x)$ from sufficiently good estimates to x. This naming system is equivalent to the δ_2 naming system in Exercise 6.2.11 of [27].

We now discuss products of spaces. Having established naming systems for spaces $X_1, \ldots, X_n$, we name a point $(p_1, \ldots, p_n) \in X_1 \times \ldots \times X_n$ by a list of n-tuples $(U_1^0, \ldots, U_n^0), (U_1^1, \ldots, U_n^1), \ldots$ such that each list $U_j^0, U_j^1, \ldots$ is a name of p_j. Let $(n, k) \mapsto \langle n, k\rangle$ be Cantor's coding of $\mathbb{N} \times \mathbb{N}$. In the case of an infinite product

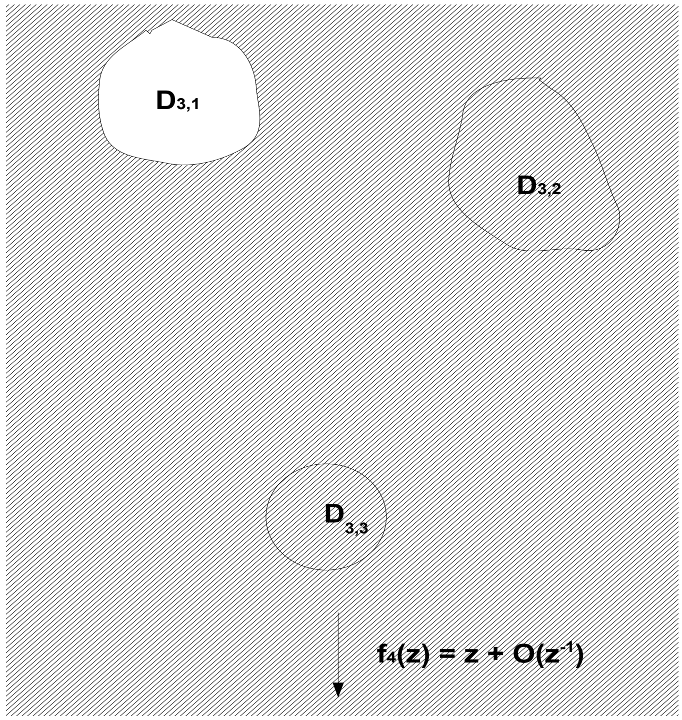


FIGURE 5. The Koebe Construction, $n = 3$, start of fourth iteration

space of the form $X \times X \times X \ldots$ we name a sequence $(p_1, p_2, \ldots)$ by a list of the form $U_0, U_1, \ldots$ where for each n, $U_{\langle n,0\rangle}, \ldots, U_{\langle n,1\rangle}, \ldots$ is a name of p_n.

A finite initial section of a name for an object, whether it be a point, a set, or a function should be regarded as an approximation to that object.

A name will be called *computable* if there is an algorithm that given any positive integer n as input, computes the n-th entry in that name. An object is called *computable* if it has a computable name.

We now discuss the model of computation of functions on these spaces. Suppose f is a function whose domain and range are each contained in one of these spaces. Suppose we have a machine (*i.e.* a procedure or a Turing machine) which after reading a finite initial section of a name of an object $p \in \text{dom}(f)$ eventually computes an entry in a name for $f(p)$. Suppose further that if a longer finite initial section of the same name is read, then the machine will compute another entry in a name for $f(p)$. And, suppose that as the machine is allowed to run, every entry which must appear in a name for $f(p)$ is eventually, after reading a sufficiently long initial segment of the name for p, computed, and that these are the only entries computed. We then say that f is *computable.* The machine should be regarded as transforming approximations to approximations in such a way that as the sequence of input approximations converges to some definite object, p, then the output approximations converge to $f(p)$. Nothing is required if $p \notin \text{dom}(f)$.

For the sake of achieving simplicity and economy, we will state our theorems in a fairly informal way with phrases like "From a name of a ... and a name of a .... it is possible to compute a name of a ...." The sense of computation here is as in the previous paragraph. Also, it is clearly intended that we do not merely have for each possible input a procedure which works on that input, but rather a procedure which works for all inputs of the specified type.

We state informally (but with sufficient precision for the present applications) some general principles of computable analysis. Precise statements and proofs can, with the exceptions noted, be found in [27].

The Principle of Type Conversion will be used to compute function-valued operators. In the present context, this means that to compute $f \in C_{open}(\subseteq \hat{\mathbb{C}})$ (or in $C_{closed}(\subseteq \hat{\mathbb{C}})$), it suffices to show that one can compute $\mathrm{dom}(f)$ and that given names of f and $z \in \mathrm{dom}(f)$ one can compute $f(z)$.

Composition of functions is a computable operator. This observation allows us to chain together sequences of computable operations on continuous spaces to form a single computable operation.

Given a name of a compact set K, a name of its complement, and a name of a real-valued f whose domain contains K, it is possible to compute the maximum and minimum values of f on K.

Given names of reals a, b and a name of a function f that has exactly one zero on $[a, b]$, we can compute a name of that zero. This statement remains true if we replace $a, b, [a, b]$ with $R, z, \overline{D_R(z)}$. For a precise statement and proof we refer to [28].

Integration is a computable operator, but differentiation is not. However, differentiation of analytic functions *is* computable. See, *e.g.*, Proposition 4.1 of [16].

To compute the limit of a sequence in $\mathbb{C}$, $\{a_n\}_{n=0}^{\infty}$, it suffices to have a name of the sequence, and a *modulus of convergence* for the sequence. This is a function $g : \mathbb{N} \to \mathbb{N}$ such that $|a_n - a_m| \leq 2^{-k}$ whenever $m, n \geq g(k)$.

3.2. **Preliminary results with regards to computation on $\hat{\mathbb{C}}$ and associated hyperspaces.** We do not include (∞, ∞) in the domain of addition so that the resulting operation will be continuous on its domain. The domain of multiplication consists of all pairs in $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ except $(0, \infty)$ and $(\infty, 0)$. The domain of division consists of all pairs in $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ except (∞, ∞) and $(0, 0)$. The proof of the following is a fairly dry technical exercise using the definitions. It can be obtained by "effectivizing" any of the usual proofs that the standard operations on $\hat{\mathbb{C}}$ are continuous.

Theorem 3.1 (**Computability of field operations).** *Addition, multiplication, and division are computable operations on* $\hat{\mathbb{C}}$.

The following results generalize results from [16].

Theorem 3.2 (**Extended Computable Open Mapping Theorem).** *From a name of a non-constant meromorphic f and a name of an open subset of its domain, U, one can compute a name of $f[U]$.*

Proof. From a name of f and a name of an open $U \subseteq \mathrm{dom}(f)$, we can compute a name of the restriction of f to U. It thus suffices to show that we can compute a name of $\mathrm{ran}(f)$.

We note that for every $z \in \mathrm{dom}(f)$ there is a subbasic neighborhood of z whose closure is contained in $\mathrm{dom}(f)$ and whose closure is either pole-free or zero-free. Using the name of f, we can build a list of the subbasic neighborhoods whose closures are zero-free and contained in $\mathrm{dom}(f)$. We can also build a list of the subbasic neighborhoods whose closures are pole-free and contained in $\mathrm{dom}(f)$. We scan these lists as we build them, and do the following. Suppose V is a pole-free neighborhood whose closure is contained in $\mathrm{dom}(f)$. We can then apply the Computable Open Mapping Theorem of Hertling [16] and begin listing all finite subbasic neighborhoods whose closures are contained in $f[V]$ as we go along. Suppose V is zero-free. Again, using Hertling's Computable Open Mapping Theorem, we can begin listing all finite subbasic neighborhoods whose closures are contained in $\frac{1}{f}[V]$. We can then also list all subbasic neighborhoods whose closures are contained in the image of $\frac{1}{z}$ on this set. We can work these neighborhoods into our output list as we go along.

What we will produce by this process is a list of subbasic neighborhoods $V_0, V_1, \ldots$ such that

$$\mathrm{ran}(f) = \bigcup_j \overline{V_j}.$$

However, it may be the case that not every subbasic neighborhood V with $\overline{V} \subseteq \mathrm{ran}(f)$ will appear in this list. What we have formed here so far is known as an *incomplete name*. However, it is quite easy to remedy the situation. Whenever subbasic neighborhoods $U_1, \ldots, U_k$ are listed, we begin working into our list all subbasic neighborhoods contained in $\bigcup_j U_j$. It follows from the compactness of $\hat{\mathbb{C}}$ that the resulting list is complete. □

Corollary 3.3. *From a name of an injective meromorphic f, we can compute a name of f^{-1}.*

Proof. We apply the Principle of Type Conversion. Suppose we are given a name of an injective meromorphic f. We can now compute a name of $\mathrm{dom}(f^{-1})$. Given a name of a $w \in \mathrm{dom}(f^{-1})$, we begin scanning the names of w and f simultaneously. Suppose we discover subbasic neighborhoods U, V such that $\overline{U} \subseteq \mathrm{dom}(f)$, $w \in V$, and $\overline{V} \subseteq f[U]$. (Such a search is now made possible by the Extended Computable Open Mapping Theorem.) Then, we can list U as a subbasic neighborhood that contains $f^{-1}(w)$. We can also begin working into our output list all subbasic neighborhoods that contain U. The resulting list is a name of $f^{-1}(w)$. □

Theorem 3.4 (Extended Computable Closed Mapping Theorem). *From a name of a meromorphic f and a name of a closed set contained in its domain, C, one can compute a name of $f[C]$.*

Proof. Begin scanning the names of f and C simultaneously. Suppose we discover a pair (V, U) in the name of f such that $V \cap C \neq \emptyset$. We can then list U as a subbasic neighborhood that hits $f[C]$. At the same time, we work into our list all subbasic neighborhoods that contain U. It follows that the resulting list is a name of $f[C]$. □

The following is an easy consequence of Hertling's Effective Riemann Mapping Theorem [16].

Lemma 3.5. *From a name of a 1-connected, non-degenerate domain D that contains ∞ but not 0, and a name of its boundary, we can compute names of C_D, ∂C_D, and f_D.*

We conclude this section with some remarks about terminology for curves. For reasons we will soon make plain, these matters must be treated more delicately than is usual in classical analysis. To begin, we define an *arc* to be the range of a continuous and injective map of a finite closed interval into $\mathbb{C}$. Such a map will be called a *parameterization* of the arc. Similarly, we define a *Jordan curve* to be the range of a continuous map γ of an interval $[a, b]$ into $\mathbb{C}$ such that $\gamma(s) \neq \gamma(t)$ whenever $s \neq t$ except when $s, t \in \{a, b\}$. Again, such a map will be called a parameterization of the curve. Unless otherwise specified, the domain of a parameterization will be assumed to be $[0, 1]$.

In classical analysis, for the sake of simplicity of exposition, one usually identifies arcs and Jordan curves with their parameterizations. In computable analysis however, we can not afford this luxury. The issue is we have adopted one convention for naming functions, and another for naming compact sets. On the one hand, from a name of a parameterization, we can compute a name of its range. However, it follows from a theorem of J. Miller [19] that we can not reverse this process.

4. Computing the sequences in the Koebe Construction

We now show that the sequences $\{D_k\}_k$, $\{f_k\}_k$, and $\{\partial D_{k,j}\}_{k,j}$ generated from the Koebe construction can be computed from the initial data $D, \partial D, n$. Our first task is to show that from these initial data we can compute the boundary components $\partial D_{0,1}, \ldots, \partial D_{0,n}$. At first sight, this may seem obvious as it may seem that we can simply look at the boundary of D and determine the component boundaries. However, our initial data do not specify the entire boundary of D (which may not even be given by Jordan curves) all at once; they merely give us a sequence of approximations to the boundary, and we must sort these into approximations to the individual components. We also need to show that we can "cover-up" components using these initial data. Mathematically, this means computing the complements of the individual components of the complement of D. All this however, can be done fairly straightforwardly. Our approach will first be to show (non-constructively), that it is possible to surround the components of the complement of D with polygons which do not intersect each other or each others' interiors and whose vertices are rational points. We call such curves *rational polygonal curves*. We then show that such curves can be discovered through a simple search procedure. This will then put us into position to compute the initial decompositions and cover-ups. Everything else follows from well-known principles of computability and computable analysis.

When γ is a smooth simple closed curve, let $Int(\gamma)$, $Ext(\gamma)$ denote the interior and exterior of γ respectively. The following has been independently observed at least by N. Müller [as reported at the Fifth International Conference on Computability and Complexity in Analysis, Hagen, Germany, 2008].

Proposition 4.1. *From a name of a parameterization of a smooth Jordan curve $\gamma : [0, 1] \to \mathbb{C}$ and a name of γ', we can compute a name of the interior of γ and a name of the exterior of γ.*

Proof. We first show that from a name $\{(U_n, V_n)\}_{n \in \mathbb{N}}$ of γ, we can compute a name of $\hat{\mathbb{C}} - \gamma$. To do so, we read these pairs while simultaneously cycling through all subbasic open sets. Whenever $n_1, \ldots, n_k$ and a subbasic U are discovered such that

$U_{n_1}, \ldots, U_{n_k}$ cover $[0,1]$ and $\overline{U}$ is disjoint from

$$\bigcup_{j=1}^{k} V_{n_j},$$

we can list U as a subbasic open set whose closure is contained in $\hat{\mathbb{C}} - \gamma$.

Let us pause here to show that this process generates a name of $\hat{\mathbb{C}} - \gamma$. It is only necessary at this point to show that every subbasic open set U whose closure is contained in $\hat{\mathbb{C}} - \gamma$ is listed by this process. Suppose U is such a set. For each $p \in \gamma$, there is a subbasic set W_p such that $p \in W_p$ and $W_p \cap \overline{U} = \emptyset$. For each $p \in \gamma$, there is a subbasic open set S_p such that $\gamma^{-1}(p) \in S_p$ and γ maps each point in $S_p \cap [0,1]$ into W_p. For each such p, we can then choose n_p such that $\gamma^{-1}(p) \in U_{n_p} \subseteq S_p$ and $V_{n_p} \subseteq W_p$. Let $U_{n_{p_1}}, \ldots, U_{n_{p_k}}$ cover $[0,1]$. Thus, $\overline{U}$ is disjoint from

$$\bigcup_{j=1}^{k} V_{n_{p_j}}.$$

And so, U is listed.

Now, using in addition the name of γ', whenever a finite subbasic U is listed by this process, we begin computing names of the maximum and minimum values of the winding number

$$\int_\gamma \frac{1}{\zeta - z} d\zeta$$

on $\overline{U}$. Of course, theses values must coincide and the common value will be zero just in case $\overline{U}$ is contained in the interior of γ. Furthermore, this common value is an integer. Whenever such a computation reveals rational numbers r_1, r_2 such that $-1 < r_1 < 0 < r_2 < 1$ and the value of the winding number on $\overline{U}$ is in (r_1, r_2), we can list U as a subbasic open set whose closure is contained in the exterior of γ. On the other hand, whenever such a computation reveals rational numbers r_1 and r_2 such that $0 \notin (r_1, r_2)$ and the value of the winding number on $\overline{U}$ is in (r_1, r_2), we can list U as a subbasic open set whose closure is contained in the interior of γ. We have thus computed names of the interior and exterior of γ. $\square$

Lemma 4.2. *Suppose $\gamma : [a,b] \to \mathbb{C}$ is a parameterization of a simple closed curve with continuous derivative. Then, for every $\epsilon > 0$, there is a parameterization of a rational, polygonal, simple closed curve γ_0 such that*

$$\begin{aligned} \max\{|\gamma(t) - \gamma_0(t)| \ : \ t \in [a,b]\} &< \epsilon \\ \sup\{|\gamma'(t) - \gamma_0'(t)| \ : \ t \in \mathrm{dom}(\gamma_0')\} &< \epsilon \end{aligned}$$

Proof. For all $s, t \in [a,b] \times [a,b]$, let

$$H(s,t) = \begin{cases} \frac{\gamma(s) - \gamma(t)}{s-t} & s \neq t \\ \gamma'(t) & s = t \end{cases}$$

We claim that H is continuous. Clearly, H is continuous at each $(s,t) \in [a,b] \times [a,b]$ such that $s \neq t$. For each such pair, there exists $c_{(s,t)}$ between s and t such that $H(s,t) = \gamma'(c_{(s,t)})$. When $s = t$, let $c_{(s,t)} = s$. For each $t_0 \in [a,b]$, $\lim_{(s,t) \to (t_0,t_0)} c_{(s,t)} = t_0$. Hence, since γ' is continuous, $\lim_{(s,t)\to(t_0,t_0)} H(s,t) = \gamma'(t_0)$. Hence, H is continuous.

Since $[a,b]\times[a,b]$ is compact, H is uniformly continuous. Let $\epsilon' = \epsilon/5$. There exists $\delta > 0$ such that $\delta < \epsilon'$ and

$$\begin{aligned} d((s,t),(s',t')) < \delta \quad &\Rightarrow \quad |H(s,t) - H(s',t')| < \epsilon' \\ |s-t| < \delta \quad &\Rightarrow \quad |\gamma(s) - \gamma(t)| < \epsilon' \end{aligned}$$

There exist $t_0, \ldots, t_k$ such that $a = t_0 < t_1 \ldots < t_k = b$, $t_1, \ldots, t_{k-1} \in \mathbb{Q}$, and $|t_{j-1} - t_j| < \frac{1}{2}\delta$ for all $j \in \{1, \ldots, k\}$. There exist $y_0, \ldots, y_k \in \mathbb{Q}\times\mathbb{Q}$ such that

$$\left|\frac{\gamma(t_{j-1}) - \gamma(t_j)}{t_{j-1} - t_j} - \frac{y_{j-1} - y_j}{t_{j-1} - t_j}\right| < \epsilon'$$

for all $j = 1, \ldots, k$. In addition, we may choose these points so that $|y_j - \gamma(t_j)| < \epsilon'$. In addition, we may choose these points so that $y_0 = y_k$.

It now follows that

$$\left|\frac{\gamma(t_{j-1}) - \gamma(t_j)}{t_{j-1} - t_j} - \gamma'(t_{j-1})\right| < \epsilon'.$$

It then follows that

$$\left|\frac{y_{j-1} - y_j}{t_{j-1} - t_j} - \gamma'(t_{j-1})\right| < 2\epsilon'.$$

It then follows that

$$\left|\frac{y_{j-1} - y_j}{t_{j-1} - t_j} - \gamma'(t)\right| < 3\epsilon' < \epsilon$$

for all $t \in [t_{j-1} - t_j]$.

Let

$$l_j(t) = \frac{t - t_{j-1}}{t_j - t_{j-1}}(y_j - y_{j-1}) + y_{j-1}$$

for all $j \in [t_{j-1}, t_j]$. Hence, l_j parameterizes the line segment $\overline{y_{j-1}y_j}$. In addition,

$$l_j'(t) = \frac{y_{j-1} - y_j}{t_{j-1} - t_j}.$$

And, when $t \in [t_{j-1}, t_j]$,

$$\begin{aligned} |l_j(t) - \gamma(t)| &\leq |l_j(t) - y_{j-1}| + |\gamma(t) - y_{j-1}| \\ &\leq |l_j(t) - y_{j-1}| + |\gamma(t) - \gamma(t_{j-1})| + |\gamma(t_{j-1}) - y_{j-1}| \\ &< |l_j(t) - y_{j-1}| + 2\epsilon' \\ &= \left|\frac{t - t_{j-1}}{t_j - t_{j-1}}(y_j - y_{j-1})\right| + 2\epsilon' \\ &= \left|\frac{t - t_{j-1}}{t_j - t_{j-1}}\right| |y_j - y_{j-1}| + 2\epsilon' \\ &\leq |y_j - y_{j-1}| + 2\epsilon' \\ &\leq |y_j - \gamma(t_j)| + |\gamma(t_j) - \gamma(t_{j-1})| + |\gamma(t_{j-1}) - y_{j-1}| + 2\epsilon' \\ &\leq 5\epsilon' = \epsilon. \end{aligned}$$

The existence of γ_0 follows immediately. □

Lemma 4.3. *Suppose D is a non-degenerate n-connected domain and $\infty \in D$. Let $C_1, \ldots, C_n$ be the components of the complement of D. Then, there exist rational polygonal curves $\gamma_1, \ldots, \gamma_n$ such that the following hold.*

(1) $\gamma_j \subseteq D$.
(2) *If* $j \neq k$, *then* $\gamma_j \cap \gamma_k = \emptyset$.
(3) $C_j \subseteq Int(\gamma_j) - \bigcup_{k \neq j} \overline{Int(\gamma_k)}$.

Proof. Let $f = f_D$ and $C = C_D$. Let $\overline{D_{r_1}(c_1)}, \ldots, \overline{D_{r_n}(c_n)}$ be the connected components of $\hat{\mathbb{C}} - C$. Let $s > 0$ be such that $\overline{D_{r_1+s}(c_1)}, \ldots, \overline{D_{r_n+s}(c_n)}$ are disjoint. Let $\gamma_j^1(t) = c_j + (r_j + s)e^{2\pi i t}$ for all $t \in [0,1]$. Let $\gamma_j^2 = f^{-1} \circ \gamma_j^1$. Since $f(\infty) = \infty$, it follows that for all $z \in D$

$$z \in Int(\gamma_j^2) \;\Leftrightarrow\; f(z) \in Int(\gamma_j^1).$$

It then also follows that $Int(\gamma_j^2) \cap Int(\gamma_k^2) = \emptyset$ when $j \neq k$. Since $Int(\gamma_j^2)$ is simply connected and $Int(\gamma_j^1) \cap D_1$ is not, it follows that $Int(\gamma_j^2)$ contains a point in the complement of D. It then follows (by connectedness), that $Int(\gamma_j^2)$ contains a connected component of the complement of D. Call this component C_j. Hence, $C_j \subseteq Int(\gamma_j^2) - \bigcup_{k \neq j} \overline{Int(\gamma_j^2)}$. Let $z_j \in C_j$ for all j. It follows (by compactness), that there exists $\epsilon > 0$ such that $D_\epsilon(\gamma_j^2(t)) \subseteq D$ for all j, t. Let $\eta(\gamma, z)$ denote the winding number of γ around z. It follows that there are rational polygonal curves $\gamma_1, \ldots, \gamma_n$ such that $\gamma_j \subseteq D$ and for all j, k

$$\left| \eta(\gamma_k^2, z_j) - \eta(\gamma_k, z_j) \right| < \frac{1}{2}.$$

Thus, $\eta(\gamma_k^2, z_j) = \eta(\gamma_k, z_j)$. Hence, $z_j \in Int(\gamma_j)$. Thus, by connectedness, $C_j \subseteq Int(\gamma_j)$ for all j. It also follows that $z_j \notin Int(\gamma_k)$ for all $k \neq j$. It then follows that $C_j \cap \overline{Int(\gamma_k)} = \emptyset$ when $j \neq k$. □

Lemma 4.4. *Suppose D is a non-degenerate n-connected domain and $\infty \in D$. Then, there exist rational rectangles $R_1, \ldots, R_n$ and rational, polygonal, simple closed curves $\gamma_1, \ldots, \gamma_n$ such that the following.*

(1) $\gamma_j \subseteq D$.
(2) *If* $j \neq k$, *then* $\gamma_j \cap \gamma_k = \emptyset$.
(3) $\overline{R_j} \subseteq Int(\gamma_j) - \bigcup_{k \neq j} \overline{Int(\gamma_k)}$.
(4) $R_j \cap \partial D \neq \emptyset$.

Furthermore, if $R_1, \ldots, R_n, \gamma_1, \ldots, \gamma_n$ are such that (1) - (4) hold, then it is possible to label the connected components of $\hat{\mathbb{C}} - D$, $C_1, \ldots, C_n$, so that $C_j \subseteq Int(\gamma_j) - \bigcup_{k \neq j} \overline{Int(\gamma_k)}$.

Proof. The existence of $R_1, \ldots, R_n, \gamma_1, \ldots, \gamma_n$ follows immediately from Lemma 4.3. So, suppose $R_1, \ldots, R_n, \gamma_1, \ldots, \gamma_n$ are such that (1) - (4) hold. It then follows that $Int(\gamma_j)$ contains a point of the complement of D. It then follows (by connectedness) that $Int(\gamma_j)$ contains a connected component of the complement of D, C_j. It then follows (again, by connectedness) that $C_j \subseteq Int(\gamma_j) - \bigcup_{k \neq j} \overline{Int(\gamma_k)}$. □

A domain is said to be *finitely connected* if its complement has finitely many connected components.

Lemma 4.5 (**Decomposition and Cover-up Lemma).** *From a name of a finitely connected, non-degenerate domain D that contains ∞, a name of its boundary, and the number of its boundary components, we can compute names of the individual boundary components as well as names of the complements of the individual components of $\hat{\mathbb{C}} - D$.*

Proof. We first search for rational rectangles $R_1, \ldots, R_n$ and curves $\gamma_1, \ldots, \gamma_n$ as in Lemma 4.4. As we read the given name of ∂D, we extract n lists from it. These lists need not be mutually exclusive. The entries in the j-th list are the subbasic neighborhoods that contain a subbasic neighborhood whose closure is entirely contained in the interior of γ_j. These lists then form names for the boundary components of D. At the same time, we can form n other lists as follows. Put into the j-th list all subbasic neighborhoods whose closures are either contained in D or the exterior of γ_j. The results are names for the complements of the components of the complement of D. □

Theorem 4.6. *Let D be a non-degenerate, finitely connected domain that contains ∞ but not 0. Let D_k, $D_{k,j}$, and f_k be as in the description of the Koebe Construction in section 2. Then, from a name of D, a name of its boundary, and the number of its boundary components, one may compute names of the sequences $\{D_k\}_k$, $\{\partial D_{k,j}\}_{k,j}$, and $\{f_k\}_k$.*

Proof. This follows from what has been shown and the use of primitive recursion in defining these sequences. (See, *e.g.* Theorem 3.1.7 of [27].) □

5. Some computations with harmonic functions

Theorem 5.1 (**Computable conjugation).** *From a name of a harmonic function, u, and names of z_0, R such that $\overline{D_R(z_0)} \subseteq \mathrm{dom}(u)$, we may compute a name of a local harmonic conjugate of u with domain $D_r(z_0)$.*

Proof. We first translate z_0 to the origin. Allow u to also denote the resulting function. Given $z \in D_R(0)$, we first compute r such that $|z| < r < R$. We can then compute

$$\tilde{u}(z) = \int_0^{2\pi} u(re^{i\theta}) \frac{re^{-i\theta} - re^{i\theta}\overline{z}}{|re^{i\theta} - z|^2} d\theta.$$

Thus, $\tilde{u}$ is the harmonic conjugate of u on $D_R(0)$ that vanishes at 0. (See, for example, page 178 of [11].) We now translate 0 back to z_0. Allow $\tilde{u}$ to also denote the resulting function.

We have shown that from names of u, R, z_0, z, we can compute $\tilde{u}(z)$. It then follows from Type Conversion that from names of u, R, z_0, we can compute $\tilde{u}$. □

Again, it has been amply demonstrated that differentiation is not a computable operator. (See, *e.g.* [20] and Theorem 6.4.3 of [27].) However, differentiation of analytic functions is computable, and the class of harmonic functions enjoys this property too.

Theorem 5.2. *From a name of a harmonic function, u, we may compute a name of $u'|_{\mathbb{C}}$. If, in addition, we are given a name of a continuously differentiable parameterization of a simple closed curve $\Gamma \subseteq \mathrm{dom}(u) \cap \mathbb{C}$ and a name of its derivative, we can compute a name of $\frac{\partial u}{\partial n}$ on Γ.*

Proof. Given names of u and $z \in \mathrm{dom}(u) \cap \mathbb{C}$, we first read these names until we find a rational rectangle R that contains z and whose closure is contained in $\mathrm{dom}(u)$. This then allows us to compute the radius of a closed disk D centered at z that is contained in R. By Theorem 5.1, we can compute a harmonic conjugate of u on the interior of D, $\tilde{u}$. Let $f = u + i\tilde{u}$. We can then compute f'. By an elementary computation, $\frac{\partial u}{\partial x} = Re(f')$ and $\frac{\partial u}{\partial y} = Re(if')$.

If our curve Γ is parameterized by $t \mapsto (x(t), y(t))$, then we may compute $\frac{\partial u}{\partial n}$ by the equation

$$\frac{\partial u}{\partial n} = \left(\frac{\partial u}{\partial x} y'(t) - \frac{\partial u}{\partial y} x'(t)\right) |x'(t) + iy'(t)|^{-1}.$$

(See, for example, pages 71 and 72 of [7].) □

We now advance towards computing solutions to Dirichlet problems on finitely connected domains. We first compute them on the disk. Let $|dz|$ denote the differential of arc length.

Lemma 5.3. *Given names of smooth arcs $\gamma_1, \ldots, \gamma_n$ and their derivatives such that $\partial\mathbb{D} = \gamma_1 + \ldots + \gamma_n$, and given names of continuous real-valued functions $f_1, \ldots, f_n$ such that $\gamma_j = \operatorname{dom}(f_j)$, we can compute a name of the harmonic function u on $\mathbb{D}$ defined by the boundary data*

$$f(\zeta) = \begin{cases} f_j(\zeta) & \zeta \in \gamma_j, \zeta \neq \gamma_j(0), \gamma_j(1) \\ \max_j \max f_j & \textit{otherwise.} \end{cases}$$

In addition we can compute the extension of u to $\overline{\mathbb{D}}$ except at the endpoints of the arcs $\gamma_1, \ldots, \gamma_n$.

Proof. Let u be the solution to the resulting Dirichlet problem on $\mathbb{D}$. For $z \in \mathbb{D}$, we use the Poisson Integral Formula

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\theta \;\; z \in \mathbb{D}. \tag{5.1}$$

(See, for example, Theorem I.1.3 of [10].) In the case under consideration, we have

$$u(z) = \sum_j \frac{1}{2\pi} \int_{\gamma_j} f_j(\zeta) \frac{1 - |z|^2}{|\zeta - z|^2} |d\zeta|.$$

Since integration is a computable operator, this shows we can compute u on $\mathbb{D}$.

Since we are given $f_1, \ldots, f_n$, it might seem immediate that we can now compute the extension of u to $\overline{\mathbb{D}}$ except at the endpoints of $\gamma_1, \ldots, \gamma_n$. However, it is not possible to determine from a name of a point $z \in \overline{\mathbb{D}}$ if $z \in \partial\mathbb{D}$. To see what the difficulty is, and to lead the way towards its solution, we delve a little more deeply into the formalism. Suppose we are given a name of a $z \in \overline{\mathbb{D}}$, p. As we read p, it may be that at some point we find a subbasic neighborhood R whose closure is contained in $\mathbb{D}$. In this case, we can just use equation (5.1). However, if we keep finding subbasic neighborhoods that intersect $\partial\mathbb{D}$, then at some point we must commit to an estimate of $u(z)$. If we guess $z \in \partial\mathbb{D}$, then later this guess and this resulting estimate may turn out to be incorrect. We face a similar problem if we guess $z \in \mathbb{D}$. The heart of the matter then is to estimate the value of $u(\zeta)$ when ζ is near z and in $\mathbb{D}$. This can be done by effectivizing one of the usual proofs that $\lim_{\zeta \to z} u(\zeta) = f(z)$ when z is between the endpoints of a γ_j. To begin, fix rational numbers $\alpha \in [-\pi, \pi]$, $2\pi > \delta > 0$, and $0 < \rho < 1$. Let

$$S(\rho, \delta, \alpha) =_{df} \{re^{i\theta} \mid \rho < r \leq 1 \;\wedge\; |\theta - \alpha| < \delta/2\}.$$

We now write the solution to the Dirichlet problem on the disk in a slightly different way that considered previously in this proof. To this end, let $P_r(\theta)$ be the *Poisson kernel*,

$$Re\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right).$$

It is fairly well-known, that if $\zeta \in \partial\mathbb{D}$ and $z \in \mathbb{D}$, then

$$\frac{1-|z|^2}{|\zeta - z|^2} = \text{Re}\left(\frac{\zeta + z}{\zeta - z}\right).$$

At the same time, if $z = re^{i\theta}$ and $\zeta = e^{i\theta'}$, then

$$Re\left(\frac{\zeta + z}{\zeta - z}\right) = P_r(\theta - \theta').$$

Let f be a function on $\partial\mathbb{D}$ such that for each arc γ_j $f(\zeta) = f_j(\zeta)$ whenever ζ is a point in γ_j besides one of its endpoints. It then follows that when $0 < r < 1$,

$$u(re^{i\theta_1}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})P(\theta_1 - \theta)d\theta.$$

We can compute a rational number M such that

$$M > \max_k \max \text{ran}(f_k).$$

Suppose α is such that for some j, $e^{i\alpha}$ is in γ_j but is not an endpoint. From the given data, we can enumerate all such α, j. We cycle through all such α, j as we scan the name of z. Fix a rational number $\epsilon > 0$. From ϵ and the given data, one can compute a rational $\delta_{\alpha,\epsilon} > 0$ such that the arc

$$\{e^{i\theta} \ : \ |\theta - \alpha| \leq \delta_{\alpha,\epsilon}\}$$

is contained in $\text{ran}(\gamma_j)$ and such that

$$\frac{\epsilon}{3} \quad > \quad \max\{|f_j(e^{i\theta}) - f_j(e^{i\alpha})| \ : \ |\theta - \alpha| \leq \delta_{\alpha,\epsilon}\}.$$

We claim we can then compute $\rho_{\alpha,\epsilon}$ such that

$$\frac{\epsilon}{3M} \quad > \quad \max\{P_r(\theta) \ : \ |\theta| \geq \frac{1}{2}\delta \ \wedge \ \rho_{\alpha,\epsilon} \leq r \leq 1\}.$$

We postpone the computation of $\rho_{\alpha,\epsilon}$ so that we can reveal our intent. Namely, we claim that $|u(\zeta) - f(e^{i\alpha})| \leq \epsilon$ when $\zeta \in S(\rho_{\alpha,\epsilon}, \delta_{\alpha,\epsilon}, \alpha)$. For, let $\zeta \in S(\rho_{\alpha,\epsilon}, \delta_{\alpha,\epsilon}, \alpha)$, and write ζ as $re^{i\theta_1}$. Hence, $\rho < r \leq 1$, and we can choose θ_1 so that $|\theta_1 - \alpha| < \delta_{\alpha,\epsilon}/2$. If $r = 1$, then there is nothing more to do. So, suppose $r < 1$. For convenience, abbreviate $\delta_{\alpha,\epsilon}$ by δ. It then follows that

$$\begin{aligned} |u(\zeta) - f(e^{i\alpha})| \quad \leq \quad & \frac{1}{2\pi}\int_{|\theta-\alpha|\geq\delta} |f(e^{i\theta}) - f(e^{i\alpha})|P(\theta_1 - \theta)d\theta \\ & + \frac{1}{2\pi}\int_{|\theta-\alpha|<\delta} |f(e^{i\theta}) - f(e^{i\alpha})|P(\theta_1 - \theta)d\theta. \end{aligned}$$

Suppose $|\theta - \alpha| \geq \delta$. Since $|\theta_1 - \alpha| < \delta/2$, $|\theta_1 - \theta| \geq \delta/2$ and so $P_r(\theta_1 - \theta) < \epsilon/3M$. Hence, the first term in the preceding sum is at most $2\epsilon/3$. At the same time, by our choice of $\delta_{\alpha,\epsilon}$, it follows that the second term is no larger than

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\epsilon}{3} P_r(\theta_1 - \theta)d\theta \leq \frac{\epsilon}{3}.$$

Hence, $|u(\zeta) - f(e^{i\alpha})| \leq \epsilon$.

Hence, as we scan the name of z, if we encounter a rational rectangle R and ϵ, α such that $S(\rho_{\alpha,\epsilon}, \delta_{\alpha,\epsilon})$ contains R, then we can list any subbasic neighborhood that contains $[f(e^{i\alpha}) - \epsilon, f(e^{i\alpha}) + \epsilon]$. It follows that if we only read neighborhoods that intersect the boundary, then we will write a name of $f(z)$ on the output tape.

We conclude by showing how to compute $\rho_{\alpha,\epsilon}$. Let $\delta' = \frac{1}{2}\delta$. The key inequality is

$$P_r(\theta) \leq P_r(\delta')$$

when $\delta' \leq |\theta| \leq \pi$. This is justified by Proposition 2.3.(c) on page 257 of [2]. Since $e^{i\delta'} \neq 1$, $P_1(\delta')$ is defined, and in fact is 0. We can compute $P_r(\delta')$ as a function of r on $[0, 1]$. We can thus compute $\rho_{\alpha,\epsilon}$ as required. □

In order to compute solutions to Dirichlet problems in the finitely connected case, we will need to be able to transfer solutions in the 1-connected case while preserving boundary data. This leads to the following. (A proof of the classical Carathéodory Theorem can be found in [12], Chapter 2, Section 3, Theorem 4.)

Theorem 5.4 (**Weak Computable Carathéodory Theorem**). *Given names of parameterizations of smooth arcs $\gamma_1, \ldots, \gamma_n$ and their derivatives, if $\gamma_1 + \ldots + \gamma_n$ is a Jordan curve, then we can compute a conformal map of the interior of this curve onto $\mathbb{D}$ as well as the homeomorphic extension to the boundary. We can compute the inverse map on $\overline{\mathbb{D}}$ as well.*

Proof. Let D be the interior of $\gamma =_{df} \gamma_1 + \ldots + \gamma_n$. We can thus compute a conformal map ϕ of D onto $\mathbb{D}$. We also use ϕ to denote the extension to $\overline{D}$. Our strategy is to first show we can compute ϕ on ∂D. We then show that we can compute the inverse on $\partial\mathbb{D}$. Let u, v be the real and imaginary parts, respectively, of ϕ^{-1}. We can then use Lemma 5.3 to compute u, v on $\overline{\mathbb{D}}$ (since solutions to Dirichlet problems are unique). This allows us to compute ϕ^{-1} on $\overline{\mathbb{D}}$. We then compute ϕ on $\overline{D}$.

We use Type Conversion to compute ϕ on ∂D. We first show how to compute ϕ between the endpoints of each γ_j. Suppose we are given a name of a z_0 between the endpoints of γ_j. We can then compute t_0 such that $\gamma_j(t_0) = z_0$.

Let A be the circle with center 0 and radius 1/2. Let $A' = \phi^{-1}[A]$. Since $A \subseteq \mathbb{D}$, we can compute names of A' and its complement. We can then compute $R > 0$ such that the circle of radius R centered at z_0 does not intersect A'.

We can also compute the line orthogonal to the tangent line to γ_j at z_0. We can then compute a neighborhood of z_0 in which the tangent lines to γ_j are not parallel to this orthogonal line. It follows that in this neighborhood, γ_j does not cross this line except at z_0. So, we can compute $t_1 < t_0 < t_2$ such that $\gamma(t)$ does not cross this orthogonal line at any $t \in [t_1, t_2]$. Select rational $t_1' \in (t_1, t_0)$ and rational $t_2' \in (t_0, t_2)$. Compute a positive lower bound δ on

$$\min\{|\gamma(t) - z_0| \ : \ t \leq t_1 \ \vee \ t \geq t_2\}.$$

Compute w_0 on this orthogonal line such that $w_0 \in D$ and $|w_0 - z_0| < R, \delta$. It follows that γ does not cross $\overline{w_0 z_0}$ except at z_0.

Suppose w_1 is a point on $\overline{w_0 z_0}$ besides z_0. From w_1, we can compute $r = |w_1 - z_0|$. We can then compute m such that

$$m^2 > \frac{(2\pi)^2}{\log(R) - \log(r)}.$$

Note that the right side of this inequality approaches 0 as r approaches 0 from the right. Let T_2 be the line $x = Re(\phi(w_1))$. Compute α such that $\phi(w_1) = |\phi(w_1)|e^{i\alpha}$. We will proceed under the assumption that T_2 hits A. For, we can apply the following construction and argument to $\psi = e^{-i\alpha}\phi$. Result will be an upper bound on $|\psi(w_1) - \psi(z_0)| = |\phi(w_1) - \phi(z_0)|$. Let T_1 be the line $x = Re(\phi(w_1)) + m$, and let T_3 be the line $x = Re(\phi(w_1)) - m$. We can thus assume m is small enough so that the lines T_1, T_3 both intersect A. Let p_1 be a point where one of these lines intersects $\partial\mathbb{D}$ and $\overline{p_1\phi(w_1)}$ does not hit A. Let $M = |p_1 - \phi(w_1)|$. We claim that $|\phi(w_1) - \phi(z_0)| < 2M$. For, suppose otherwise. Then, when ϕ is applied to $\overline{w_1 z_0}$, then resulting arc hits T_2 and one of T_1, T_3. Without loss of generality, suppose it hits T_1. Let $S_k = \phi^{-1}[T_k]$ for $k = 1, 2$. Hence, S_1, S_2 hit $\overline{w_1 z_0}$. Hence, every circle centered at z_0 and whose radius is between r and R inclusive hits S_1 and S_2. This puts us in position to use the "Length-Area Trick". Let $C_{r'}$ denote the circle with radius r' centered at z_0. Fix $r \leq r' \leq R$. The curves S_1, S_2 have positive minimum distance from each other. It follows that there are points $z_{1,r'}$ and $z_{2,r'}$ on $C_{r'}$ that belong to S_1, S_2 respectively and such that no points on these curves appear on $C_{r'}$ between $z_{1,r'}$ and $z_{2,r'}$. (We do not claim that we can compute such points; this is not necessary to prove that our estimate is correct.) Let $K_{r'}$ denote the arc on $C_{r'}$ from $z_{1,r'}$ to $z_{2,r'}$. Hence, for some $\theta_{1,r'}, \theta_{2,r'}$

$$\begin{aligned} |\phi(z_{1,r'}) - \phi(z_{2,r'})| &= \left| \int_{K_{r'}} \phi'(z) dz \right| \\ &\leq \int_{\theta_{1,r'}}^{\theta_{2,r'}} |\phi'(z)| r' d\theta \end{aligned}$$

On the other hand, $\phi(z_{1,r'})$ is on T_1 and $\phi(z_{2,r'})$ is on T_2. Hence,

$$m \leq \int_{\theta_{1,r'}}^{\theta_{2,r'}} |\phi'(z)| r d\theta.$$

At the same time, by the Schwarz Integral Inequality,

$$m^2 \leq \int_{\theta_{1,r'}}^{\theta_{2,r'}} |\phi'(z)|^2 d\theta \int_{\theta_{1,r'}}^{\theta_{2,r'}} (r')^2 d\theta.$$

Whence

$$\begin{aligned} \frac{m^2}{r'} &\leq r' \int_{\theta_{1,r'}}^{\theta_{2,r'}} |\phi'(z)|^2 d\theta \int_{\theta_{1,r'}}^{\theta_{2,r'}} d\theta \\ &\leq 2\pi r' \int_{\theta_{1,r'}}^{\theta_{2,r'}} |\phi'(z)|^2 d\theta. \end{aligned}$$

If we now integrate both sides of this inequality with respect to r' from r to R, we obtain

$$m^2[\log(R) - \log(r)] \leq 2\pi \int_r^R \int_{\theta_{1,r'}}^{\theta_{2,r'}} |\phi'(z)|^2 r' d\theta dr'.$$

It follows from the Lusin Area Integral (see *e.g.* Lemma 13.1.2, page 386, of [13]) that this double integral is no larger than 2π. Hence,

$$m^2 \leq \frac{(2\pi)^2}{\log(R) - \log(r)}.$$

This is a contradiction. So, $|\phi(z_0) - \phi(w_1)| < 2M$.

As r approaches 0 from the right, $\phi(w_1)$ approaches $\phi(z_0)$ and so m, M can be chosen so as to approach 0. It follows that we can now generate a name of $\phi(z_0)$.

We have now computed ϕ on ∂D except at the endpoints of $\gamma_1, \ldots, \gamma_n$. However, using betweeness and connectedness relationships, we can now also locate the images of these endpoints with arbitrary precision. Since ϕ is injective, for each $z_0 \in \partial\mathbb{D}$, $\phi - z_0$ has a unique 0 on $\partial\mathbb{D}$, and it follows from the remarks in Section 3 that we can compute ϕ^{-1} on $\partial\mathbb{D}$. It now follows as noted in the introduction to this proof that we can compute ϕ^{-1} on $\overline{\mathbb{D}}$. For each $z_0 \in \overline{D}$, $\phi^{-1} - z_0$ has a unique zero. Hence, we can compute ϕ on $\overline{D}$. □

We do not know if we can eliminate the assumption of piecewise differentiability

Theorem 5.5 (Computable Solution of Dirichlet Problems). *Given a name of a Jordan domain D and names of smooth $\gamma_1, \ldots, \gamma_n$ and their derivatives, if $\gamma_1, \ldots, \gamma_n$ are the distinct boundary components of D, and if we are also given a name of a continuous $f : \partial D \rightarrow \mathbb{R}$, then we can compute a solution of the corresponding Dirichlet problem. Furthermore, we can compute an extension of this solution to $\overline{D}$.*

Proof. We first do some conformal pre-configuring of our domain. First, we compute a finite $z_0 \in D$. Let

$$h_1(z) = \frac{1}{z - z_0}.$$

Set $D_1 = h_1[D]$. Hence, $\infty \in D_1$. Let $\gamma_j^1 = h_1[\gamma_j]$. Let $f_1 = f \circ h_1^{-1}$. Let

$$D_1' = D_1 \cup \bigcup_{j=2}^{n} \overline{Int(\gamma_j^1)}.$$

By the Boundary Decomposition and Cover Up Lemma, we can compute D_1' from the given data. Note that $\partial D_1' = \gamma_1^1$. We can now compute $f_{D_1'}$ and $C_{D_1'}$. Let h_2 be the restriction of $f_{D_1'}$ to D_1. Let $\gamma_1^2 = \partial C_{D_1'}$. Let $\gamma_j^2 = f_{D_1'}[\gamma_j^1]$ for $j = 2, \ldots, n$. Let $D_2 = h_2[D_1]$. It follows from Theorem 5.4 that we can computably transfer the boundary data from D_1 to D_2 and thusly compute a function f_2. We can compute the center w_0 and radius r_0 of γ_1^2 from the given data (by first using $\partial C_{D_1'}$ to compute three points on γ_1^2). Hence, we can compute the inversion

$$h_3(z) = \frac{r_0^2}{z - w_0}.$$

Let $D_3 = h_3[D_2]$, $\gamma_1^3 = \gamma_1^2$, and $\gamma_j^3 = h_3[\gamma_j^2]$ for $j = 2, \ldots, n$, *etc.*. Hence, D_3 is contained in the interior of γ_1^3 which is a circle. Let $f_3 = f_2 \circ h_3^{-1}$.

We now apply the ideas in the proof of Theorem II.1.1 and Exercise II.2 of [10]. To this end, we wish to compute a piecewise linear curve σ such that $D_3 - \sigma$ is simply connected. Furthermore, we will ensure that σ hits γ_1^3 at one point only. We will also ensure that it hits each of $\gamma_2^3, \ldots, \gamma_{n-1}^3$ at exactly two points, and γ_n^3 at exactly one point. In fact, these properties ensure $D_3 - \sigma$ is simply connected.

To get started, compute a point $w_{1,1}$ on γ_1^3. Compute also distinct points on γ_j^3, $w_{j,1}$ and $w_{j,2}$, for each $j \in \{2, \ldots, n-1\}$. Compute a point $w_{n,1}$ on γ_n^3. Finally, compute a point z_1 in the exterior of γ_1^3 and a point z_2 in D_3.

Fix $j \in \{2, \ldots, n-1\}$. We compute a piecewise linear curve μ_j from $w_{j,1}$ to $w_{j,2}$ that hits γ_2^3 at $w_{j,1}$ and $w_{j,2}$ only. To do so, we first compute a point $a \in Int(\gamma_j^3)$ such that $\overline{w_{j,1}a}$ hits γ_j^3 at $w_{j,1}$ only. This can be done using the technique used to construct w_0 in the proof of Theorem 5.4. We then compute a point $b \in Int(\gamma_j^3)$

such that $\overline{w_{j,2}b}$ hits γ_j^3 at $w_{j,2}$ only and does not hit $\overline{w_{j,1}a}$. We then compute a piecewise linear curve in $Int(\gamma_j^3)$ from a to b.

Having computed these curves, we then compute a piecewise linear curve μ_n in $\overline{Int(\gamma_n^3)}$ from $w_{n,1}$ to a point in $Int(\gamma_n^3)$ that hits γ_n^3 at $w_{n,1}$ only.

We now compute a piecewise linear curve λ_0 from z_1 to z_2 that hits γ_1^3 at exactly one point. We then compute a piecewise linear curve λ_1 in $\overline{D_3}$ from z_2 to $w_{2,1}$ that hits ∂D_3 at $w_{2,1}$ only and does not hit λ_0. Then, for each $j \in \{2, \ldots, n-1\}$, we compute a piecewise linear curve λ_j in $\overline{D_3}$ from $w_{j,2}$ to $w_{j+1,1}$ that hits ∂D_3 at $w_{j,2}$ and $w_{j+1,1}$ only and that does not hit $\lambda_0 + \lambda_1 + \mu_2 + \lambda_2 + \ldots + \mu_{j-1} + \lambda_{j-1}$. Let $\sigma = \lambda_0 + \lambda_1 + \mu_2 + \lambda_2 + \ldots + \lambda_{n-1} + \mu_n$.

We now similarly compute a piecewise linear curve σ' such that $D_3 - \sigma'$ is simply connected and $\sigma' \subseteq D_2 - \operatorname{ran}(\sigma)$.

We can now compute $\Omega_1 =_{df} D_3 - \sigma$ and $\Omega_1' =_{df} D_3 - \sigma'$.

Let z_n be the terminal point of σ in $Int(\gamma_n^3)$. Similarly, let z_n' be the terminal point of σ' in $Int(\gamma_n^3)$. Compute an analytic branch ψ_1 of $\sqrt{\frac{z-z_n}{z-z_1}}$ on Ω_1 and an analytic branch ϕ_1 of $\sqrt{\frac{z-z_n'}{z-z_1'}}$ on Ω_1'. Compute a conformal map of $\psi_1[\Omega_1]$ onto $\mathbb{D}$, ψ_2. Compute also a conformal map of $\phi_1[\Omega_1']$ onto $\mathbb{D}$, ϕ_2.

Now, let $f_3' = f_3 + 2|\min \operatorname{ran}(f_3)|$ so that $f_3' > 0$. We now generate a sequence of harmonic functions $u_1 \geq u_1' \geq u_2 \geq u_2' \geq \ldots \geq 0$ as in the proof of Theorem II.1.1 of [10] with f_3' in place of f. It follows from Lemma 5.3 and the Weak Computable Carathéodory Theorem that the functions in this sequence can be computed from the given information. We now use the suggestion of Exercise II.2 of [10] to show that $u =_{df} \lim u_n$ can be computed from the given information. We first introduce a little notation. Let g_n, g_n' be the boundary data used to define u_n, u_n' respectively. Let $\phi = \phi_2\phi_1$ and $\psi = \psi_2\psi_1$. Let:

$$\begin{aligned} A &= \psi[\partial\Omega_1] \\ B &= \psi[\sigma] \\ A' &= \phi[\partial\Omega_1'] \\ B' &= \phi[\sigma'] \end{aligned}$$

Let:

$$\begin{aligned} u_{n,2} &= u_n\psi^{-1} \\ u_{n,2}' &= u_n'\phi^{-1} \\ g_{n,2} &= g_n\psi^{-1} \\ g_{n,2}' &= g_n'\phi^{-1} \end{aligned}$$

Let $P(z,\zeta)$ abbreviate

$$Re\left(\frac{\zeta + z}{\zeta - z}\right).$$

Let:

$$\begin{aligned}
H_1(z) &= \frac{1}{2\pi}\int_A P(\psi(z),\zeta)g_{1,2}(\zeta)|d\zeta| \\
H_2(z) &= \frac{1}{2\pi}\int_{A'} P(\phi(z),\zeta)g'_{1,2}(\zeta)|d\zeta| \\
h(z) &= H_1(z) + \frac{1}{2\pi}\int_B P(\psi(z),\zeta)H_2(\psi^{-1}(\zeta))|d\zeta| \\
K(z,\zeta_1) &= \int_B P(\psi(z),\zeta)P(\phi\psi^{-1}(\zeta),\zeta_1)|d\zeta|
\end{aligned}$$

Note that $K(z,\zeta) > 0$.

We now show that

$$u_{n+1}(z) = h(z) + \frac{1}{(2\pi)^2}\int_{B'} K(z,\zeta_1)u_n(\phi^{-1}(\zeta_1))|d\zeta_1|. \tag{5.2}$$

We first note that

$$\begin{aligned}
u_{n+1}(z) &= u_{n+1,2}(\psi(z)) \\
&= \frac{1}{2\pi}\int_A P(\psi(z),\zeta)g_{n+1,2}(\zeta)|d\zeta| + \frac{1}{2\pi}\int_B P(\psi(z),\zeta)g_{n+1,2}(\zeta)|d\zeta|
\end{aligned}$$

However, for $\zeta \in A$, $g_{n+1,2}(\zeta) = g_{1,2}(\zeta)$. It follows that

$$u_{n+1}(z) = H_1(z) + \frac{1}{2\pi}\int_B P(\psi(z),\zeta)g_{n+1,2}(\zeta)|d\zeta|.$$

Let $\zeta \in B$. Then, $g_{n+1,2}(\zeta) = g_{n+1}(\psi(\zeta))$. But, by the definition of B, $\psi^{-1}(\zeta) \in \sigma$. By the definition of u_{n+1} therefore, $g_{n+1}(\psi^{-1}(\zeta)) = u'_n(\psi^{-1}(\zeta))$. Hence, we may write

$$\int_B P(\psi(z),\zeta)g_{n+1,2}(\zeta)|d\zeta| = \int_B P(\psi(z),\zeta)u'_n(\psi^{-1}(\zeta))|d\zeta|.$$

We now expand $u'_n(\psi^{-1}(\zeta))$ in terms of u_n. Fix $z \in \Omega_1$. We have

$$\begin{aligned}
u'_n(z) &= u'_{n,2}(\phi(z)) \\
&= \frac{1}{2\pi}\int_{A'} P(\phi(z),\zeta)g'_{n+1,2}(\zeta)|d\zeta| + \frac{1}{2\pi}\int_{B'} P(\phi(z),\zeta)g'_{n+1,2}(\zeta)|d\zeta|
\end{aligned}$$

But, for $\zeta \in A'$, $g'_{n+1,2}(\zeta) = g'_{1,2}(\zeta)$. Hence,

$$u'_n(z) = H_2(z) + \frac{1}{2\pi}\int_{B'} P(\psi(z),\zeta)g'_{n+1,2}(\zeta)|d\zeta|.$$

When $\zeta \in B'$, $g'_{n+1,2}(\zeta) = g_{n+1}(\phi^{-1}(\zeta))$ and $\phi^{-1}(\zeta) \in \sigma'$. On the other hand, when $w \in \sigma'$, $g_{n+1}(w) = u_n(w)$. Hence, for all $\zeta \in B'$, $g'_{n+1,2}(\zeta) = u_n(\phi^{-1}(\zeta)$. So, we have

$$\int_{B'} P(\phi(z),\zeta)g'_{n+1,2}(\zeta)|d\zeta| = \int_{B'} P(\phi(z),\zeta)u_n(\phi^{-1}(\zeta))|d\zeta|.$$

So,

$$u'_n(z) = H_2(z) + \frac{1}{2\pi}\int_{B'} P(\phi(z),\zeta)u_n(\phi^{-1}(\zeta))|d\zeta|.$$

Hence,

$$u_n'(\psi^{-1}(z)) = H_2(\psi^{-1}(z)) + \frac{1}{2\pi}\int_{B'} P(\phi\psi^{-1}(z),\zeta)u_n(\phi^{-1}(\zeta_1))|d\zeta_1|.$$

Pulling all this together, we obtain

$$u_{n+1}(z) \quad = \quad h(z) + \frac{1}{(2\pi)^2}\int_B P(\psi(z),\zeta)\int_{B'} P(\phi\psi^{-1}(\zeta),\zeta_1)u_n(\phi^{-1}(\zeta_1))|d\zeta_1||d\zeta|$$

At the same time,

$$\begin{aligned}
& \int_B P(\psi(z),\zeta)\int_{B'} P(\phi\psi^{-1}(\zeta),\zeta_1)u_n(\phi^{-1}(\zeta_1))|d\zeta_1||d\zeta| \\
= & \int_B\int_{B'} P(\psi(z),\zeta)P(\phi\psi^{-1}(\zeta),\zeta_1)u_n(\phi^{-1}(\zeta_1))|d\zeta_1||d\zeta| \\
= & \int_{B'}\int_B P(\psi(z),\zeta)P(\phi\psi^{-1}(\zeta),\zeta_1)u_n(\phi^{-1}(\zeta_1))|d\zeta||d\zeta_1| \\
= & \int_{B'}\left(\int_B P(\psi(z),\zeta)P(\phi\psi^{-1}(\zeta),\zeta_1)|d\zeta|\right)u_n(\phi^{-1}(\zeta_1))|d\zeta_1| \\
= & \int_{B'} K(z,\zeta_1)u_n(\phi^{-1}(\zeta_1))|d\zeta_1|.
\end{aligned}$$

Equation (5.2) now follows.

Let $\parallel v \parallel_\infty^{\sigma'}$ denote the L^∞ norm of v on σ'. We now show that $u_1, u_2, \ldots$ forms a Cauchy sequence in the L^∞ norm on σ'. We do so in such a way as to compute a modulus of convergence for this sequence. Fix $z \in \sigma'$ and $n \geq 2$. We have

$$|u_{n+1}(z) - u_n(z)| = \frac{1}{(2\pi)^2}\left|\int_{B'} K(z,\zeta_1)[u_n(\phi^{-1}(\zeta_1)) - u_{n-1}(\phi^{-1}(\zeta_1))]|d\zeta_1|\right|.$$

Since $\phi^{-1}(\zeta_1) \in \sigma'$ for all $\zeta_1 \in B'$, it follows that

$$|u_{n+1}(z) - u_n(z)| \leq \frac{1}{(2\pi)^2} \parallel u_n - u_{n-1} \parallel_\infty^{\sigma'} \int_{B'} K(z,\zeta_1)|d\zeta_1|.$$

Using Fubini's Theorem, we can rewrite the latter integral as

$$\int_{B'} K(z,\zeta_1)|d\zeta_1| = \int_B P(\psi(z),\zeta)\left(\int_{B'} P(\phi\psi^{-1}(\zeta),\zeta_1)|d\zeta_1|\right)|d\zeta|.$$

We examine the integral

$$\int_{B'} P(\phi\psi^{-1}(\zeta),\zeta_1)|d\zeta_1| \tag{5.3}$$

Recall that

$$P(z,\zeta) = \frac{1-|z|^2}{|\zeta - z|^2}.$$

For all $\zeta \in B$, $\phi\psi^{-1}(\zeta)$ is bounded away from B', and we can compute a positive lower bound on this distance. It follows as in the proof of Lemma 5.3 that we can compute (5.3) as a function of ζ on B and hence can compute its maximum on B (the only issue is to use the techniques of this proof to estimate what happens near the boundary), m_2. Since

$$2\pi = \int_{B'} P(\phi\psi^{-1}(\zeta),\zeta_1)|d\zeta_1| + \int_{A'} P(\phi\psi^{-1}(\zeta),\zeta_1)|d\zeta_1|$$

it follows that $m_2 < 2\pi$. So, we obtain that

$$\int_{B'} K(z,\zeta_1)|d\zeta_1| \leq m_2 \int_B P(\psi(z),\zeta)|d\zeta|.$$

By a similar analysis, we can compute the maximum of the latter integral as z ranges over σ', m_1, and $m_1 < 2\pi$. Let

$$c = \frac{m_1 m_2}{(2\pi)^2}.$$

Hence, $c < 1$. We have now shown that

$$\| u_{n+1} - u_n \|_\infty^{\sigma'} \leq c \| u_n - u_{n-1} \|_\infty^{\sigma'}.$$

Whence it follows that

$$\| u_{n+1} - u_n \|_\infty^{\sigma'} \leq c^{n-1} \| u_2 - u_1 \|_\infty^{\sigma'}.$$

We can compute a rational number M such that $M > \max \mathrm{ran}(f_3')$. Since $u_2 \leq u_1 \leq \max f_3'$, it follows that $\| u_{n+1} - u_n \|_\infty^{\sigma'} \leq c^{n-1} M$. So, if $m, n > k$, then

$$\begin{aligned} \| u_m - u_n \|_\infty^{\sigma'} &\leq \sum_{j=k}^{\infty} \| u_{j+1} - u_j \|_\infty^{\sigma'} \\ &\leq M \frac{c^{k-1}}{1-c}. \end{aligned}$$

Now, suppose $z \in \Omega_1$. For $m, n > 1$,

$$\begin{aligned} |u_m(z) - u_n(z)| &\leq \frac{1}{(2\pi)^2} \| u_m - u_n \|_\infty^{\sigma'} \int_{B'} K(z,\zeta_1)|d\zeta_1| \\ &\leq \| u_m - u_n \|_\infty^{\sigma'}. \end{aligned}$$

It now follows that $u_1, u_2, \ldots$ is a Cauchy sequence in the L^∞ norm on Ω_1 and that we can compute a modulus of convergence for this sequence on Ω_1. It then follows from continuity considerations that this same modulus of convergence applies to all of D_3. Hence, we can compute u from the given data.

As we read a name of $z \in \overline{D_3}$, if we ever discover a rectangle whose closure is contained in D_3, then we can proceed with our algorithm for computing u on D_3. Suppose at some point this has not happened and we discover a rectangle R that hits γ_1^3 and no other boundary component. Fix R. Note that $R \cap D_3$ is a Jordan domain since γ_1^3 is a circle. Inside $R \cap D_3$, u is the harmonic function determined by the boundary data

$$h(\zeta) = \begin{cases} f_3'(\zeta) & \zeta \in \gamma_1^3 \\ u(\zeta) & \zeta \in \partial R \end{cases}$$

Suppose we later discover a rational rectangle R' that hits γ_1^3 and whose closure is contained in R. Compute a conformal map ϕ_3 of $R \cap D_3$ onto $\mathbb{D}$. By the Weak Computable Carathéodory Theorem, we can compute the homeomorphic extension of ϕ_3 to the closure of $R \cap D_3$. Let ϕ_3 denote this extension as well. Set $h_3 = h \circ \phi_3^{-1}$. We can assume R small enough so that $\overline{R} \cap \gamma_1^3$ is connected. We can compute h_3 on the arc $\phi_3[\overline{R} \cap \gamma_1^3]$. We can also compute a rational number M such that $M > \max \mathrm{ran}(f)$. Hence, by the Lindelöf Maximum Principle (see *e.g.* Lemma I.1.1 of [10]), $M > u(z_0)$ for all $z_0 \in \overline{R \cap D_3}$. Hence, we can use the technique at the end of the proof of Lemma 5.3 to compute a suitable rational rectangle R_1 such that $u(z) \in R_1$.

On the other hand, suppose instead that we later discover a rational rectangle R that hits γ_j^3 and no other boundary component where $j \neq 1$. We can compute a conformal map ϕ_4 of the exterior of γ_j^3 onto the exterior of $\mathbb{D}$. If R is small enough, we can then compute a rational rectangle R' that contains $\phi_4[\overline{R} \cap \overline{D_3}]$ and that does not hit the images under ϕ_4 of the other boundary components. We then compute

$$C =_{df} \phi_4^{-1}[\partial R' \cap (\hat{\mathbb{C}} - \mathbb{D}) \ \cup \ (R' \cap \partial\mathbb{D})]$$

which is a Jordan curve that does not hit the other boundary components. We can also compute the interior of C. On the interior of C, u is the harmonic function defined by the boundary data

$$h_C(\zeta) = \begin{cases} u(\zeta) & \zeta \notin \gamma_j^3 \\ f_3'(\zeta) & \zeta \in \gamma_j^3 \end{cases}$$

We can now proceed as in the case where R hits γ_1^3 only.

It now follows that we can compute u on $\overline{D_3}$. We can then transfer this solution back to $\overline{D}$ and f. □

Theorem 5.6 (**Computable Harmonic Extension**). *Given a name of a domain D, a name of a harmonic $u : \overline{D} \to \mathbb{R}$, and names of conformal $f_1, \ldots, f_n$ such that*

- $f_j(\infty) = \infty$,
- $\overline{\mathbb{D}} \subseteq \mathrm{dom}(f_j)$,
- $\gamma_j =_{df} f_j[\partial\mathbb{D}]$ *is a boundary component of D on which u is zero, and*
- $\gamma_1, \ldots, \gamma_n$ *are distinct,*

we can compute a neighborhood of $D \cup (\bigcup_j \gamma_j)$, D', and a harmonic extension of $u|_D$ to D'.

Proof. Fix $j \in \{1, \ldots, n\}$ for the moment. From the name of f_j, we can compute a name of $\mathrm{dom}(f_j)$. We can then compute a name of $\exp^{-1}[\mathrm{dom}(f_j)]$. (See *e.g.* Theorem 6.2.4.1 of [27].) We can then compute a covering of the line segment from $-i\pi$ to $i\pi$ by rational rectangles $R_1, \ldots, R_k$ whose closures are contained in $\exp^{-1}[\mathrm{dom}(f_j)]$. Let r be the minimum distance from a vertical side of one of these rectangles to the y-axis. Hence, r is computable from $R_1, \ldots, R_k$. It now follows that for each $-\pi \leq y \leq \pi$, each point on the line segment from $-r + iy$ to $r + iy$ is contained in $\exp^{-1}[\mathrm{dom}(f_j)]$. We can now compute a positive rational number $r_{0,j}$ such that $r_{0,j} < e^{-r}$. Let A_j be the annulus centered at the origin and with inner radius $1 - r_{0,j}$ and outer radius $1 + r_{0,j}$. Hence, $\overline{A_j} \subseteq \mathrm{dom}(f_j)$. Let g be the reflection map for $\partial\mathbb{D}$. *i.e.*

$$g(z) = \frac{1}{\overline{z}}.$$

Hence, A_j is closed under reflection.

Let

$$D' = D \ \cup \ \bigcup_{j=1}^{n} f_j[A_j].$$

We can choose $r_{0,1}, \ldots, r_{0,n}$ so that $f_1[A_1], \ldots, f_n[A_n]$ are pairwise disjoint. It follows from the Extended Computable Open Mapping Theorem (Theorem 3.2) that D' can be computed from the given data. We define v on D' as follows. Given

$z \in D'$, if $z \in D$, then let $v(z) = u(z)$. Otherwise, there exists unique j such that $z \in f_j[A_j]$, and we let

$$v(z) \quad = \quad u f_j g f_j^{-1}(z). \tag{5.4}$$

It follows from Schwarz Reflection (see *e.g.* Theorem 4.12 of [1]), that v is harmonic on D'. It only remains to show we can compute v from the given data. Here is how we do this. Given a name p of a $z \in D'$, we read p until we find a subbasic neighborhood R such that either $\overline{R} \subseteq D$ or $\overline{R} \subseteq f_j[A_j]$. In the first case, we simply compute $u(z)$. In the second case, we can use (5.4). □

6. Computation of harmonic measure and capacity

Let D be a Jordan domain with boundary curves $\Gamma_1, \ldots, \Gamma_n$. For each $z \in D$, define $\omega(z, \Gamma_j, D)$ to be the value at z of the solution to the Dirichlet problem with boundary data

$$f(\zeta) = \begin{cases} 1 & \zeta \in \Gamma_j \\ 0 & \text{otherwise} \end{cases}$$

The function ω is called *harmonic measure.*

The following is an immediate consequence of Theorem 5.5 and the definition of harmonic measure. We state it as a theorem because of its importance for what follows.

Theorem 6.1 (**Computability of Harmonic Measure).** *Given a name of D and names of parameterizations of $\Gamma_1, \ldots, \Gamma_n$ as above, if $\Gamma_1, \ldots, \Gamma_n$ are smooth and we are also given names of the derivatives of these parameterizations, then we can compute names of the corresponding harmonic measure functions. Furthermore, we can compute their extensions to $\overline{D}$.*

Theorem 6.2 (**Computability of the Riemann Matrix).** *Given the same initial data as in Theorem 6.1, we can compute a name of the period of the harmonic conjugate of $\omega(\cdot, \Gamma_i, D)$ around Γ_j.*

Proof. The period of the harmonic conjugate of $\omega(\cdot, \Gamma_i, D)$ around Γ_j is defined to be

$$\frac{1}{2\pi} \int_{\Gamma_j} \frac{\partial \omega(\zeta, \Gamma_i, D)}{\partial n} |d\zeta|.$$

Given what has been covered already, nothing more is necessary to justify the conclusion. □

Let D be an n-connected domain such that $\infty \in D$ and the boundary components of D are Jordan curves. Fix $a \notin \overline{D}$. Let h be the solution to the Dirichlet problem with boundary data

$$f(\zeta) = \log \left| \frac{1}{\zeta - a} \right|.$$

Let $G(z, \infty) = \log |z - a| + h(z)$. G is called a *Green's function* for D. Now, let E be the complement of D. Let

$$\gamma(E) = \lim_{z \to \infty} G(z, \infty) - \log |z - a|.$$

$\gamma(E)$ is called the *Robin's constant* of E. We define the *capacity* of E to be $e^{-\gamma(E)}$. The capacity of $\partial E = \partial D$ is defined to be that of E. If f is a conformal map of D onto D' of the form $z + a + O(z^{-1})$, then ∂D and $\partial D'$ have the same capacity.

The following uses the spectacular result of Ransford and Rostand [24] on the computation of capacity. The statement of this theorem is quite involved and covers a great many cases. Rather than state it here, we will just refer to the pertinent parts of it in the following proof and refer the reader to their paper for the complete statement.

Theorem 6.3 (**Computability of Capacity**). *Given a name of a finitely connected Jordan domain D that contains ∞, a name of its boundary, and the number of its boundary components, we can compute a name of the capacity of its boundary.*

Proof. For every positive integer n, let

$$\Delta_n = \{(t_1, \ldots, t_n) \mid t_j \geq 0, \sum_j t_j = 1\}.$$

Hence, Δ_n is a computable compact subset of $\mathbb{R}^n$. Now, for every $n \times n$ matrix h, define $M(h)$ to be

$$\min_{s \in \Delta_n} \max_{t \in \Delta_n} \sum_{i,j} h_{i,j} s_i t_j.$$

It follows that $h \mapsto M(h)$ is computable.

Let n be the number of boundary components of D. We can then compute the sequences generated by the Koebe construction. Let $\{D_k\}_k$, $\{\partial D_{k,j}\}_{k,j}$, $\{f_k\}_k$, and $\{g_k\}_k$ denote these sequences as in Section 2.

Fix $k > n$. Note that ∂D and ∂D_k have the same capacity. Each boundary component of D_k is the image of a conformal map normalized at ∞ on the boundary of a disk. From this, we can compute the diameter of each boundary component of D_k as well as the minimal distance between these boundary components.

Fix $0 < r < \min\{\operatorname{diam}(\partial D_{k,1}), \ldots, \operatorname{diam}(\partial D_{k,n}), d/2\}$ where d is the minimal distance between any two boundary components of D_k. Using the parameterizations of $\partial D_{k,1}, \ldots, \partial D_{k,n}$, we can compute $w_1, \ldots, w_m \in \partial D_k$ so that $\overline{D_r(w_1)}, \ldots, \overline{D_r(w_m)}$ cover ∂D_k. Hence, if w_j and w_k lie on distinct boundary components of D_k, then $\overline{D_r(w_j)}$ and $\overline{D_r(w_k)}$ are disjoint. It now follows from Theorem 5.3.3(a) of [23] that the capacity of $\overline{D_r(w_j)} \cap \partial D_k$ is at least $r/4$. So, let $\delta = r/4$, and define $m \times m$ matrices a and b by the equations

$$\begin{aligned} a_{i,j} &= \log \frac{1}{\operatorname{diam}(\overline{D_r(w_i)} \cup \overline{D_r(w_j)})} \\ b_{i,j} &= \log \frac{1}{\max\{\delta, d(\overline{D_r(w_i)}, \overline{D_r(w_j)})\}} \end{aligned}$$

It follows that we can compute a, b from the given information. Hence, we can also compute $M(a)$ and $M(b)$. Theorem 3.1 of [24] states that

$$M(a) \leq \log \frac{1}{c} \leq M(b)$$

where c is the capacity of ∂D_k. It now follows from the remarks following Theorem 3.2 of the same paper that $M(b) - M(a)$ approaches 0 as r approaches 0. It now follows that we can compute the capacity of ∂D_k. $\square$

7. Computability of the constants associated with a circular domain

Proposition 7.1. *From a name of a non-degenerate, finitely connected domain D, a name of its boundary, and the number of its boundary components, we can compute an upper bound on ρ_D.*

Proof. First, compute rational polygonal curves as in the proof of the Boundary Decomposition and Cover-up Lemma. We can then compute $r > 0$ so that $D_r(0)$ contains all of these curves. So, f_D is analytic and univalent in the domain $\hat{\mathbb{C}} - \overline{D_r(0)}$.

We claim that $\rho_D \leq 2r$. Although this is somewhat well-known, we include a proof here for the sake of completeness. Let $F(\xi) = f_D(r\xi)$. Hence, F is univalent outside $\overline{\mathbb{D}}$. Suppose c is not in the range of f_D. Then, the function defined by

$$h(z) = \frac{1}{F(\frac{1}{z}) - c} = \frac{1}{\frac{r}{z} + ... - c} = \frac{1}{r}z + \frac{c}{r^2}z^2 + ... = \frac{1}{r}[z + \frac{c}{r}z^2 + ...]$$

is univalent in the unit disk. We can then apply a well known theorem on univalent functions (see, *e.g.*, Theorem 7.7 of Section 14.7 of [3]) to the function $z + \frac{c}{r}z^2 + ...$ to obtain $|\frac{c}{r}| \leq 2$. Hence the boundary of the image of $|\zeta| > r$ under f_D is contained in the disk $|z| < 2r$ and we have the estimate $\rho_D \leq 2r$. □

If C is a circular domain, then let $r_{min}(C)$ be the smallest radius of a disk in $\hat{\mathbb{C}} - C$, and let $d_{min}(C)$ be the smallest distance between points in distinct components of $\hat{\mathbb{C}} - C$. Let $r_{max}(C)$ be the largest radius of a disk in $\hat{\mathbb{C}} - C$. When D is a non-degenerate, finitely connected domain, let

$$\begin{aligned} r_{min}(D) &= r_{min}(C_D) \\ d_{min}(D) &= d_{min}(C_D) \end{aligned}$$

As we shall see, obtaining positive lower bounds on these numbers will allow us to obtain a positive lower bound on δ_D and an upper bound on μ_D which is less than one. We will also need an upper bound on $r_{max}(D)$, which is trivial at this point.

Proposition 7.2. *From a name of a non-degenerate, finitely connected domain that contains ∞, a name of its boundary, and the number of its boundary components, we may compute an upper bound on $r_{max}(D)$.*

Proof. This follows from our bound on ρ_D. □

In order to obtain a positive lower bound on $r_{min}(D)$, we will use Thurman's result on the distortion of capacity. This is Theorem 1 of [25]. The following is the necessary part of that Theorem for our purposes.

Theorem 7.3 (Upper bound on distortion of capacity; Thurman, 1994). *Suppose D is an unbounded Jordan domain with boundary curves $\Gamma_1, \ldots, \Gamma_m$. Suppose A is the union of the first k of these curves, and that B is the union of the other curves. Set $v_i = \omega(\infty, \Gamma_{i+k}, D)$. Let $p_{i,j}$ be the period of the harmonic conjugate of $\omega(\cdot, \Gamma_{i+k}, D)$ about Γ_{j+k}. Finally, set $P = (p_{i,j})$ and define $(c_1, \ldots, c_{m-k})$ by*

$$(c_1, \ldots, c_{m-k}) = P^{-1}v.$$

Then, if f is a conformal map of D onto D' of the form $z + O(z^{-1})$, the capacity of $f[A]$ is no larger than the capacity of ∂D multiplied by

$$\exp\left(-\sum_{i=1}^{m-k} c_i v_i\right).$$

This inequality is sharp, and extremal domains exist. (Hence, this multiplier is necessarily less than 1.)

The remainder of Thurman's result explicitly characterizes the extremal domains.

Theorem 7.4. *From the same input data as in Proposition 7.2, if the boundary components of D are Jordan curves, then we can compute a positive lower bound on $r_{min}(D)$.*

Proof. We first compute an upper bound R on ρ_D. Let $D_1 = \frac{1}{2R}D$. It follows that

$$f_D(z) = 2Rf_{D_1}(\frac{1}{2R}z).$$

It follows that $C_{D_1} \subseteq D_{1/2}(0)$. It thus suffices to compute a positive lower bound of $r_{min}(D_1)$.

Let Γ_1 be a boundary component of D_1. Let Γ_2 be the union of the other boundary components of D_1. Let E_j be the image of Γ_j under f_{D_1}. Let $E = \partial C_{D_1}$. It follows from Theorems 6.1 and 5.2 that we can compute the matrix P in Theorem 7.3. It now follows from 7.3 that we can compute an upper bound on the capacity of E_2 which is also less than the capacity of E. (The computation of matrix inverses in the context of Type-Two Effectivity is covered in [29].) We can hence compute a positive lower bound on the Robin's constant of E_2, $\gamma(E_2)$, which is also larger than the Robin's constant of E. It follows that we can compute rational numbers r_1, r_2 such that

$$\frac{1}{\gamma(E_2)} < r_1 < r_2 < \frac{1}{\gamma(E)}.$$

However, by Lemma III.7.5 of [10],

$$\frac{1}{\gamma(E)} \leq \frac{1}{\gamma(E_2)} + \frac{1}{\gamma(E_1)}.$$

Hence, $0 < r_2 - r_1 < 1/\gamma(E_1)$. It now follows that we can compute a positive lower bound on the capacity of E_1. But, this is just the radius of E_1. So, by multiplying this lower bound by $2R$, we obtain a lower bound on the radius of one of the circles in ∂C_D. By repeating this process with each boundary component of D_1, we obtain a lower bound on $r_{min}(D)$. □

The following Lemmas summarize the relations between μ_D and $d_{min}(D)$. Both will be used later.

Lemma 7.5. *From a name of a non-degenerate, finitely connected domain D, a name of its boundary, the number of its boundary components, and a positive lower bound on $d_{min}(D)$, we can compute a lower bound on μ_D^{-1} that is larger than 1.*

Proof. Let C_1, C_2 be two boundary components of C_D. Let z_j, r_j be the center and radius respectively of C_j. Without loss of generality, suppose $z_1, z_2 \in \mathbb{R}$ and $z_1 < z_2$. Suppose $0 < \delta' < d_{min}(D)$ and $r' > r_{max}(D)$. We have

$$z_1 + r_1 + d_{min}(D) \leq z_2 - r_2.$$

Let ϵ be a real such that $1 < \epsilon < 1 + \delta'/4r'$. Hence,

$$\epsilon - 1 < \frac{\delta'}{4r_1}.$$

It now follows that

$$z_1 + \epsilon r_1 \quad < \quad z_1 + r_1 + d_{min}(D)/4. \tag{7.1}$$

It similarly follows that

$$z_2 - \epsilon r_2 \quad > \quad z_2 - r_2 - d_{min}(D)/4. \tag{7.2}$$

Since $d_{min}(D) \leq (z_2 - r_2) - (z_1 + r_1)$, it follows that the right side of 7.2 is larger then the left side of 7.1. Hence, $1 < \epsilon < \mu_D^{-1}$. The Lemma follows. ☐

Lemma 7.6. *From a name of a non-degenerate, finitely connected domain D, a name of its boundary, the number of its boundary components, and a lower estimate on μ_D^{-1} that is larger than 1, we can compute a positive lower bound on $d_{min}(D)$.*

Proof. Let C_j, z_j, r_j be as in the proof of Lemma 7.5. Again, without loss of generality, we suppose z_1, z_2 are real and $z_1 < z_2$. Suppose $1 < \epsilon < \mu_D^{-1}$. On the other hand,

$$z_1 + \epsilon r_1 < z_2 - r_2.$$

We can rewrite this inequality as

$$z_1 + r_1 + (\epsilon - 1)r_1 < z_2 - r_2.$$

From which we can conclude

$$(\epsilon - 1)r_{min}(D) < (z_2 - r_2) - (z_1 + r_1).$$

Since the choice of C_1, C_2 was arbitrary, we can assume that $d_{min}(D) = (z_2 - r_2) - (z_1 + r_1)$. It follows that $(\epsilon - 1)r_{min}(D) < d_{min}(D)$. ☐

The following allows us to use positive lower estimates on $r_{min}(D)$ and $d_{min}(D)$ to obtain a positive lower estimate on δ_D.

Lemma 7.7. *If D is a non-degenerate, finitely connected domain that contains ∞, then*

$$\delta_D \geq \frac{1}{\frac{1}{r_{min}(D)} + \frac{1}{d_{min}(D)}}.$$

Proof. Since the distance between two points is invariant under translation and rotation in the complex plane, assume that we are given two circles C_1 and C_2 with radii r_1 and r_2 and centers on the real line. The center of C_1 is at the origin O and the center of C_2 is at ζ_2, $\zeta_2 > r_2$. For our purposes we assume that $r_1 + r_2 < \zeta_2$. Let $d_{12} = \zeta_2 - (r_1 + r_2)$. Let $\eta_1 = r_1$. Suppose C_2 intersects the positive real axis at points $\eta_2 < \zeta_2$ and at $\eta_2' > \zeta_2$. A reflection of C_2 in C_1 will result in a circle C_2^1 lying inside C_1, whose diameter is on the positive part of the real line between O and η_1. Hence the reflection of the point η_2, which we will denote η_2^1, will be the point closest to the circle C_1. Denote this distance between η_1 and η_2^1 by $d_{12,1}$. We know that

$$\eta_2 \cdot \eta_2^1 = r_1^2.$$

Since $\eta_2 = r_1 + d_{12}$, then

$$d_{12,1} = r_1 - \eta_2^1 = r_1 - \frac{r_1^2}{\eta_2} = r_1 - \frac{r_1^2}{r_1 + d_{12}} = \frac{r_1 d_{12}}{r_1 + d_{12}} = \frac{1}{\frac{1}{r_1} + \frac{1}{d_{12}}}.$$

Hence $d_{12,1}$ decreases as either r_1 or d_{12} or both decrease.

We are given n circles with $r_1, \cdots, r_n$ radii. Therefore there are $n(n-1)$ distances $d_{12}, \cdots, d_{1n}, d_{23}, \cdots, d_{2n}, \cdots, d_{(n-1)n}$ between the n circles, which corresponds to the fact that there are n circles and $n-1$ reflections into each circle, or $n(n-1)$ reflections. Let

$$r_0 = \min\{r_1, \cdots, r_n\}$$

and

$$d_0 = \min\{d_{12}, \cdots, d_{1n}, d_{23}, \cdots, d_{2n}, \cdots, d_{(n-1)n}\}.$$

Then

$$\delta_D \geq \frac{1}{\frac{1}{r_0} + \frac{1}{d_0}}.$$

□

Let ρ be the hyperbolic pseudometric on $\mathbb{D}$:

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right|.$$

The following is Theorem 0.6 of [14] and will be our chief weapon when obtaining a positive lower estimate on $d_{min}(D)$. It is a generalization of the Schwarz-Pick Lemma.

Theorem 7.8. *Let A, A' be bounded Jordan domains such that $A \subseteq \mathbb{D}$ and $A' \supseteq \mathbb{D}$. Let Ω, Ω' be obtained by removing at most countably many points and closed disks from A, A' respectively. Suppose f is a conformal map of Ω' onto Ω. Then, the restriction of f to $\mathbb{D}$ satisfies the inequality*

$$\rho(f(z), f(w)) \leq \rho(z, w).$$

We will now stop short of actually obtaining a positive lower estimate on $d_{min}(D)$ from $D, \partial D$. Rather, we will show that we can do this for the n-connected case once we have solved our main problem for the $(n-1)$-connected case. This then reduces the whole problem to the doubly connected case which will be handled in the next section.

Lemma 7.9. *Suppose $n \geq 3$. Suppose also that from a name of an $(n-1)$-connected, non-degenerate domain D that contains ∞ and a name of its boundary, we can compute f_D and C_D. Then, from a name of an n-connected, non-degenerate domain D that contains ∞ and a name of its boundary, we can compute a positive lower estimate on δ_D.*

Proof. We first perform some computable manipulations on D. Again, compute the sequences in the Koebe construction, and label them as in Section 2. Hence, $D_{1,1}$ is a circle. Reflect D_1 into $D_{1,1}$, and let E_1 be the resulting domain. We then take the exterior of the $(n-1)$ components inside $D_{1,1}$, and map this domain onto an $(n-1)$-connected circular domain, E_2. The boundary curve $\partial D_{1,1}$ is distorted in the process and is no longer a circle. Call the new curve Γ_1. We can now compute a linear map that maps this curve and its interior into $\mathbb{D}$. Let E_3 denote the image of this map on the domain whose boundary components are the curve Γ_1 and the $(n-1)$ circles inside Γ_1.

We now perform some operations on C_D, and we do not claim that we can compute their results in advance from the given data. Let $C_1, \ldots, C_n$ be the boundary components of C_D. There is a unique linear map $f_1(z) = az + b$ that maps C_1 onto $\partial\mathbb{D}$. Let C' denote the image of f_1 on C_D, and let C'_j denote the image of f_1 on C_j. Now, invert C' into $\mathbb{D}$. Let C'' denote the resulting domain, and let C''_j denote the image of this inversion on C'_j. Hence, $C''_1 = \partial\mathbb{D}$.

We can compute the minimal pseudohyperbolic distance between the circles inside Γ_1. By Theorem 7.8, this bounds below the minimal pseudohyperbolic distance between any two of the circles $C''_2, \ldots, C''_n$. Call this number λ_{hyper}. We now seek to translate this into a positive lower bound on the minimal Euclidean distance between any two of the circles $C_2, \ldots, C_n$. We first seek to find a lower bound on the Euclidean distance between any two of the circles $C''_2, \ldots, C''_n$. We begin by noting that each of the circles $C'_2, \ldots, C'_n$ has radius no smaller than $r_{min}(D)/r_{max}(D)$. Hence, each of their centers is outside the circle with center 0 and radius $1 + r_{min}(D)/2r_{max}(D)$. Hence, the centers of $C''_2, \ldots, C''_n$ are all inside the circle with center 0 and radius

$$r_{inner} =_{df} \frac{1}{1 + r_{min}(D)/2r_{max}(D)}.$$

Now, without loss of generality, suppose the least Euclidean distance between any two of these circles occurs between C''_2 and C''_3. Let $z_2 \in C''_2$ and $z_3 \in C''_3$ be the points at which this least Euclidean distance occurs. It follows from elementary plane geometry that z_2, z_3 lie on the line between the centers of C''_2 and C''_3. Hence, they lie inside the circle with radius r_{inner}. Let m be the minimum of $|1 - z\overline{w}|$ over all z, w of modulus at most r_{inner}. Hence, $m > 0$, and we have

$$\lambda_{hyper} \leq \left| \frac{z_2 - z_3}{1 - z_2\overline{z_3}} \right| \leq \frac{|z_2 - z_3|}{m}.$$

So, the least Euclidean distance between C''_2 and C''_3 is no smaller than $m\lambda_{hyper}$, which we have computed. Let $\lambda_{inner} = m\lambda_{hyper}$.

We now translate this lower bound into a lower bound on the minimal Euclidean distance between any two of $C'_2, \ldots, C'_n$. We first reflect things back out of $\mathbb{D}$. We note that when $z, w \in \mathbb{D}$,

$$\left| \frac{1}{z} - \frac{1}{w} \right| = \frac{|z - w|}{|zw|} \geq |z - w| \geq \lambda_{inner}.$$

We can then compute a positive lower bound on the minimal Euclidean distance between any two of the circles $C'_2, \ldots, C'_n$. Call this lower bound λ_{outer}.

We now apply the inverse of the linear transformation which led to $C'_1, \ldots, C'_n$ from $C_1, \ldots, C_n$. If r_1 is the radius of C_1, then this transformation had the form

$$z \mapsto \frac{1}{r_1} z + b.$$

So, its inverse has the form

$$z \mapsto r_1 z + c.$$

So, the minimal Euclidean distance between any two of the circles $C_2, \ldots, C_n$ is no smaller than $r_{min}(D)\lambda_{outer}$. And, we can now compute a positive lower bound on this quantity.

By repeating this process with say C_2 in place of C_1, we can compute a positive lower bound on $d_{min}(D)$. By Lemma 7.7, we can now compute a positive lower bound on δ_D. □

8. Statements and proofs of the main theorems

Theorem 8.1. *From a name of a finitely connected, non-degenerate domain D that contains ∞ but not 0, a name of its boundary, and the number of its boundary components, we can compute names of f_D, C_D, and ∂C_D.*

Proof. In light of Lemma 7.9, it suffices to show that from a name of a 2-connected non-degenerate domain D that contains ∞ and a name of its boundary, we can compute names of f_D, C_D, and ∂C_D. This is accomplished as follows.

The *modulus* of an annulus is defined to be the ratio of its outer radius to its inner radius. The modulus of an arbitrary non-degenerate 2-connected domain is defined as the modulus of any annulus it is conformally equivalent to. Use μ here to denote the reciprocal of the modulus of D. We will need to compute μ. Unfortunately, we can not use this definition to do as this would lead to circular reasoning. We overcome this difficulty as follows.

Define an associated number μ^* (known as the *logarithmic modulus* of D) by the equation

$$\mu^* = \frac{1}{2\pi}\log \mu^{-1}.$$

It suffices to show how to compute μ^* from the given data. This is done as follows. We first compute the sequences generated by the Koebe construction. Denote these as in Section 2. Consider D_2. $D_{2,2}$ is a circle, and we can compute its center and radius, z_0, r_0, respectively. We may then compute the image of D_2 under the map

$$z \mapsto \frac{r_0}{z - z_0} - z_0.$$

Denote this image by E. Hence, $E \subseteq \mathbb{D}$. Denote the boundary components of E by E_1, E_2 with $E_1 = \partial\mathbb{D}$. Let

$$\phi(z) = \omega(z, E_1, E).$$

We can compute ϕ, an extension of ϕ to a neighborhood of $\overline{E}$, and the partials of this extension. Let ϕ denote this extension as well. It is shown in [9] that the reciprocal of μ^* is

$$\int\int_{\overline{E}} |Grad(\phi)|^2 dxdy.$$

It follows from Theorems 5.6 and 6.1 that we can compute an extension of ϕ to a neighborhood of E. We can then compute $Grad(\phi)$ on E. We can then compute an upper bound on this integral. We can then use this to compute an upper bound on μ that is smaller than 1.

Now, let A be the annulus

$$\{z \in \mathbb{C} \mid \mu < |z| < 1\}.$$

We compute a conformal map of E onto A. The construction we use is known as the *Komatu Construction* (see [8]). We first compute a point z_1 in the interior of

E_2. We then compute an automorphism of the disk which exchanges z_1 and 0. Let ψ denote this map. Let F_0 denote the resulting region, and let $F_{0,1}$, $F_{0,2}$ denote the corresponding boundary curves with $F_{0,2} = \partial\mathbb{D}$.

Let h_1 be the conformal map of the exterior of $F_{0,1}$ onto the exterior of $\partial\mathbb{D}$. Let F_1 be the image of h_1 on F_0, and let $F_{1,1}$ and $F_{1,2}$ denote the corresponding boundary curves. Let h_2 be the conformal map of the interior of $F_{1,2}$ onto $\mathbb{D}$ such that $h_2(0) = 0$. Call the image of h_2 on F_1 F_2 and let $F_{2,1}$ and $F_{2,2}$ be the corresponding boundary curves. We now repeat this process and obtain maps $h_3, h_4, \ldots$. Let $g_m = h_m \circ h_{m-1} \circ \ldots \circ h_1$.

It is well-known that this sequence of maps converges to a conformal map of F_0 onto A, g. In [8], Satz 9.3, it is shown that the error can be estimated by

$$|g_m(z) - g(z)| \leq 13\mu^{2m}.$$

(It was later shown that the constant 13 can be reduced to 8.) It follows that we can compute g from the given information.

Let A_1 be the image of A under the inversion $\frac{1}{z}$, and let $A_{1,1}, A_{1,2}$ denote its boundary curves with $A_{1,1} = \partial\mathbb{D}$. Let $w_1 = \frac{1}{g(z_1)}$. Compute R so that $\overline{D_R(w_1)} \subseteq A_1$. Let U denote the image of A_1 under the inversion

$$z \mapsto \frac{R^2}{z - w_1}.$$

By composing all of these maps, we obtain and compute a conformal map H of D_2 onto U which maps ∞ to itself. It follows from Theorem 5.4 that we can compute ∂U. This allows us to compute μ_U. We claim that $\mu_U = \mu_D$. For, it follows from Theorem IX.36 of [26] that $f_D \circ H^{-1}$ is a fractional linear transformation. Since this transformation maps ∞ to ∞, it is in fact simply linear. It then follows that $\mu_U = \mu_D$. Hence, we can now calculate μ_D. It then follows from Lemma 7.6 that we can calculate a positive lower bound on $d_{min}(D)$. In light of Theorem 7.4 and Lemma 7.7, this now puts us in possession of suitable estimates on the constants used in Henrici's bound on the error in the Koebe Construction. We can thus now compute $f_D, C_D, \partial C_D$. (In the same manner that we determined an upper bound on ρ_C, we can determine how to map neighborhoods of ∞ to neighborhoods of ∞.)

The theorem is thus now proven. □

Corollary 8.2. *From a name of a finitely connected, non-degenerate domain D, a name of its boundary, the number of its boundary components, and a name of a $z_0 \in D$, we can compute a name of a circular domain C and a conformal mapping of D onto C, f, such that $f(z_0) = \infty$.*

Corollary 8.3. *Given a name of a finitely connected, smooth Jordan domain D that contains ∞, and names of its boundary components and their derivatives, one can compute the homeomorphic extension of f_D to $\overline{D}$.*

Proof. Let $g_0, g_1, \ldots$ be as in Section 2. It follows from Theorem 5.4 that we can compute the homeomorphic extension of each g_k to $\overline{D}$. By continuity, the error bound in Theorem 2.1 holds for these extensions as well. □

9. A reversal

The following is from Hertling [16].

Lemma 9.1. *Suppose U is a proper, open, connected subset of $\mathbb{C}$, and f is a conformal map on U. Then, for all $z \in U$*

$$\frac{1}{4}|f'(z)|d(z, \mathbb{C} - U) \leq d(f(z), f(U)) \leq 4|f'(z)|d(z, \mathbb{C} - U).$$

Our first goal is to show that any neighborhood that hits the boundary of D must hit it at a finite point (when D is a non-degenerate, finitely connected domain). We will accomplish this with a little point-set topology. Recall that a point c of a connected space K is said to be a *cut point* of K if $K - \{c\}$ is not connected.

Lemma 9.2. *Suppose Z is a connected space with no cut point. Suppose O' is a subset of Z such that O' and $Z - O'$ contain at least two points. Then $\partial O'$ contains at least two points.*

Proof. If $O' \subseteq Z$ and $\partial O' = \{q\}$, then $Z - \{q\} = [O' - \{q\}] \cup [(Z - O') - \{q\}]$. On the other hand, $\partial O' = \partial(Z - O')$. Hence, O' and $[(Z - O') - \{q\}]$ are open, disjoint, and non-empty. Thus, q is a cut point of Z- a contradiction. □

Lemma 9.3. *Suppose D is a non-degenerate n-connected domain and $O \subseteq \hat{\mathbb{C}}$ is an open set such that $O \cap \partial D \neq \emptyset$. Then there exists a point $p \in \mathbb{C}$ such that $p \in O \cap \partial D$.*

Proof. Assume that $O \subseteq \hat{\mathbb{C}}$ is open and $\infty \in (\partial D) \cap O$. We can assume $O = \hat{\mathbb{C}} - \overline{R}$ for some rational rectangle R. Let K_∞ be the component of $\hat{\mathbb{C}} - D$ that contains ∞. By connectedness, $K_\infty - \{\infty\}$ is not bounded in $\mathbb{C}$. Therefore, $\mathbb{C} \cap (O \cap K_\infty)$ contains at least two points. At the same time, $\mathbb{C} - (O \cap K_\infty)$ contains at least two points. The result then follows from Lemma 9.2. □

Theorem 9.4. *From a name of a finitely connected, non-degenerate domain D that contains ∞ but not 0, the number of its boundary components, a name of f_D, and a name of ∂C_D, one may compute the boundary of D.*

Proof. By the Extended Computable Open Mapping Theorem, we can compute a name of C_D from these data. It only remains to compute names of the boundary of D and C_D. To this end, suppose R, R', w_0 are such that

$$\begin{aligned} R, R' &\in \mathbb{Q} \\ w_0 &\in D \cap \mathbb{Q} \times \mathbb{Q} \\ R' &\geq \frac{4R}{|f'_D(w_0)|} \\ R &\geq d(f_D(w_0), \hat{\mathbb{C}} - C_D) \end{aligned}$$

The last inequality would be witnessed by the containment in $\overline{D_R(f_D(w_0))}$ of a basic neighborhood that hits C_D. Let $z_1 = f_D(w_0)$. It now follows from Lemma 9.1 that

$$\frac{1}{4}|(f_D^{-1})'(z_1)|d(z_1, \hat{\mathbb{C}} - C_D) \leq d(f_D^{-1}(z_1), \hat{\mathbb{C}} - D) \tag{9.1}$$

$$\leq 4|(f_D^{-1})'(z_1)|d(z_1, \hat{\mathbb{C}} - C_D) \tag{9.2}$$

It now follows that $D_{R'}(w_0)$ hits the boundary of C_D. So, once we discover such R, R', w_0, we can begin listing all basic neighborhoods that contain the closure of this disk as among those that hit the boundary of C_D. It follows from Lemma

9.3 that every basic neighborhood that hits the boundary of C_D contains a finite neighborhood that does. Since z_1 approaches the boundary of C_D as w_0 approaches the boundary of D, it follows from (9.2) that there are arbitrarily small disks of the form $D_{R'}(w_0)$. It now follows that we can compute a name of the boundary of C_D. It then similarly follows that we can compute a name of the boundary of D. $\square$

It is well-known that there is a computable 1-connected domain whose boundary is not computable.

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