Computing conformal maps onto canonical slit domains

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Outline

- Let $\mathbb{C}$ denote the complex plane.
- Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
- Let $\mathbb{D}$ denote the unit disk centered at 0.

Definition

1. A *domain* is an open connected subset of $\hat{\mathbb{C}}$.
2. A domain is *n-connected* if its complement has exactly n components.
3. A domain is *degenerate* if a component of its complement consists of a single point.

Definition

A domain is *Jordan* if each of its boundary components is a Jordan curve.

Recall that when $f :\subseteq \mathbb{C} \to \mathbb{C}$,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

- f is *analytic* if its domain is an open connected subset of $\mathbb{C}$ and for every $z_0 \in \text{dom}(f)$ f is differentiable in a neighborhood of z_0.
- f is *conformal* if $f'(z) \neq 0$ for all $z \in \text{dom}(f)$.

Theorem (Riemann Mapping Theorem)

Every non-degenerate 1*-connected domain is conformally equivalent to the unit disk.*

This makes the disk the *canonical domain* for 1-connected domains.

When $n > 1$, it is NOT the case that all n-connected non-degenerate domains are conformally equivalent.

This leads to canonical domain *classes*. The most common can be sorted into the

- circular domains
- slit domains (several kinds of these; will discuss later)
- polygonal domains

- We investigate effective content of results on conformal mapping within framework of TTE.
- We use representations of $\hat{\mathbb{C}}$ and its hyperspaces given by standard topology on $\hat{\mathbb{C}}$.
- Our starting point is:

Theorem (Hertling, 1999)

From a name of a 1-connected non-degenerate domain U and a name of its boundary, one can compute a name of a conformal map of U onto $\mathbb{D}$.

(*n.b.* a constructive proof of the Riemann Mapping Theorem appears in Bishop and Bridges [1].)

- Our goal is to generalize Hertling’s result as much as possible to domains of connectivity > 1.

Theorem (Andreev, Daniel, McNicholl 2008)

From a name of a finitely connected, non-degenerate domain D, a name of its boundary, the number of its boundary components, and a name of a $z_0 \in D$, we can compute a name of a circular domain C and a conformal mapping of D onto C, f, such that $f(z_0) = \infty$.

Theorem (Ibid, 2008)

If D is bounded by smooth Jordan curves, and if we are also given names of these curves and their derivatives, then we can compute the homeomorphic extension of f to $\overline{D}$.

n.b. We name a Jordan curve Γ with a name of a parameterization that is one-to-one except at endpoints.

Dirichlet Problems Recall that a real-valued function u on a domain D is *harmonic* if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Let D be a finitely connected Jordan domain, and let f be a bounded piecewise continuous function on ∂D. The resulting *Dirichlet problem* is to find a harmonic $u_f : D \to \mathbb{R}$ such that

$$\lim_{z \to \zeta} u_f(z) = f(\zeta)$$

for all $\zeta \in \partial D$ at which f is continuous.

Theorem

Dirichlet problems always have unique solutions.

Proof can be found in Garnett and Marshall, 2005 [2].

Theorem (Andreev, Daniel, McNicholl 2008)

Given a name of a Jordan domain D and names of smooth $\gamma_1, \ldots, \gamma_n$ and their derivatives, if $\gamma_1, \ldots, \gamma_n$ are the distinct boundary components of D, and if we are also given a name of a continuous $f : \partial D \to \mathbb{R}$, then we can compute a solution of the corresponding Dirichlet problem. Furthermore, we can compute an extension of this solution to $\overline{D}$.

Theorem (Andreev, Daniel, McNicholl 2008)

From a name of a harmonic function u we can compute a name of u'.

Outline

Theorem (Computability of Slit Disk Mappings part 1)

Given names of:

- *a finitely connected, non-degenerate domain D,*
- *the number of boundary components of D,*
- *∂D and one of the boundary components of D, γ, and*
- *a point $\zeta_0 \in D$*

we can compute a conformal mapping ψ_D of D onto a slit disk domain that maps ζ_0 to 0 and γ to $\partial\mathbb{D}$.

Green's Functions Let D be a Jordan domain, and let $\zeta_0 \in D$. Let u_{ζ_0} be the harmonic function on D such that

$$u_{\zeta_0}(\zeta) = -\operatorname{Log}|\zeta - \zeta_0|$$

for all $\zeta \in \partial D$. Let

$$G_D(z, \zeta_0) = u_{\zeta_0}(\zeta) + \operatorname{Log}|z - \zeta_0|.$$

G_D is called the *Green's function* of D.

Harmonic conjugates

Definition

Let u, v be harmonic in D. v is *conjugate* to u if $u + iv$ is analytic.

Basic facts about conjugates Suppose u is harmonic in a domain D.

- Any two conjugates of u differ by a constant.
- u has a local conjugate at every $z \in D$.
- If D is simply connected, then u has a conjugate on all of D.
- If D is multiply connected, then u may not have a conjugate on all of D.

- Let s denote the arc length parameter. Let

$$\frac{\partial u}{\partial n}$$

denote the normal derivative of u.

- If v is conjugate of u, then

$$v(z) = \int_{\zeta_0}^{z} \frac{\partial u}{\partial n} ds + v(\zeta_0).$$

Harmonic measure

Theorem

If D is a Jordan domain, and if $E \subseteq \partial D$ is Borel, then there is a unique harmonic function on D, $\omega(\cdot, E, D)$, whose boundary values are given by the characteristic function of E.

Proof is in Garnett and Marshall [2] and others.
When the boundary curves of D are $\Gamma_1, \ldots, \Gamma_n$, we let $\omega_j = \omega(\cdot, \Gamma_j, D)$.

Lemma

If u is harmonic in a bounded domain D with boundary curves $\Gamma_1, \ldots, \Gamma_n$ the outermost of which is Γ_n, then there are unique $b_1(u), \ldots, b_{n-1}(u)$ such that

$$u + \sum_{j=1}^{n-1} b_j(u)\omega_j$$

has a conjugate in D.

Lemma (Andreev, McNicholl 2009)

For smooth Jordan domains, from names of Γ_j and their derivatives and a name of u, we can compute names of $b_1(u), \ldots b_{n-1}(u)$.

Since differentiation of harmonic functions is computable, we can even compute a name of the conjugate!

Proof sketch of slit disk part 1 We 'effectivize' a construction due to Max Schiffer [6].

- We first compute a conformal map f of D onto a domain D_1 bounded by analytic curves $\gamma_1, \ldots, \gamma_n$. Also compute γ_j, γ_j'.
- Using result on Dirichlet problems, we construct
  - Green's function for D_1, G, and
  - harmonic measure functions for D_1
- Using Lemma, we compute a multivalued "analytic extension" of $G(\cdot, f(\zeta_0))$, $\hat{G}$.
- $\psi_D = \exp(\hat{G} \circ f)$.

Theorem (Computability of Slit Disk Maps part 2)

When the boundary curves of D are smooth Jordan curves, and we are in addition given names of these curves and their derivatives, we can compute the continuous extension of ψ_D to $\overline{D}$.

Outline

Conformal maps onto slit annulus, circular slit, and parallel slit domains can be reduced to the slit disk maps (see *e.g.* Nehari [4]).

Outline

Let D be a bounded domain with smooth boundary curves $\Gamma_1, \ldots, \Gamma_n$. Let $f \in C(\partial D)$, and suppose $\int_{\partial D} f\, ds = 0$. The resulting *Neuman problem* is to find a harmonic function u on D such that

$$\frac{\partial u}{\partial n} = f \text{ on } \partial D \quad (1)$$

$$\int_{\partial D} u\, ds = 0 \quad (2)$$

Theorem (Computing solutions of Neuman problems)

Given names of D, f, and $\Gamma_1, \ldots, \Gamma_n$ as well as their derivatives, one can compute a name of the solution of the resulting Neuman problem.

Proof 'effectivizes' a well-known procedure for finding solutions to Neuman problems for *multiply* connected domains.

Outline

Theorem

From a name of non-degnerate, finitely connected, domain D, a name of its boundary, names of distinct $\zeta_0, \zeta_1 \in D$*, and the number of its boundary components, we can compute a name of a conformal map of D onto a radial slit domain that maps* ζ_0 *to* 0 *and* ζ_1 *to* ∞

Given Data:

1. A bounded domain D_0 bounded by Jordan curves $\Gamma_0, \ldots, \Gamma_M$ the outermost of which is Γ_0.
2. A selection of *prevertices* $\{a_k^{(j)}\}_{j=0,\ldots,M,k=1,\ldots,n_j}$ such that $a_k^{(j)} \in \Gamma_j$ and $\Gamma_j^{-1}(a_1^{(j)}) < \ldots < \Gamma_j^{-1}(a_{n_j}^{(j)})$.
3. A selection of reals in $(0, 1)$, $\{\beta_k^{(j)}\}_{j=0,\ldots,M,k=1,\ldots,n_j}$, such that

$$\sum_{k=1}^{n_j}(1 - \beta_k^{(j)}) = 2.$$

4. A $\zeta_0 \in D_0$.

There is a unique conformal map ϕ_0 and a bounded polygonal domain P (also unique) such that

- ϕ_0 maps D_0 onto P,
- ϕ_0 maps $a_k^{(j)}$ to a vertex of a boundary component of P, $p_k^{(j)}$,
- the exterior angle at $p_k^{(j)}$ is $\pi(1 - \beta_k^{(j)})$,
- $\phi_0'(\zeta_0) = 1$, and
- $\phi_0(\zeta_0) = 0$.

ϕ_0 is called a *Schwarz-Christoffel map*.

Theorem (Andreev, McNicholl, 2009)

From names of the Given Data we can compute a name of ϕ_0.

Additional results

- We can also compute the homeomorphic extension of ϕ_0 to $\overline{D_0}$.
- We can also compute maps onto unbounded polygonal domains.

Preliminary manuscript available. (Email me.)

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