Computable Aspects of Inner Functions

Timothy H. McNicholl

mcnichollth@my.lamar.edu
Department of Mathematics
Lamar University
Beaumont, Texas 77710 USA

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Outline

The aim of this talk We will discuss some results in computable complex analysis.

Let f, g, h range over analytic functions only.

Naming systems

- $\theta_<$ for set of open subsets of $\mathbb{C}$.
- ψ for set of closed subsets of $\mathbb{C}$.
- $[\rho^2]^\omega$ for set of countably infinite sequences of complex numbers.
- $[\rho^2 \to \rho^2]_{\mathbb{D}}$ for the set of continuous $f : \mathbb{D} \to \mathbb{C}$
- δ_{open} for set of continuous $f :\subseteq \mathbb{C} \to \mathbb{C}$ with open domain. This representation is defined by:

$$\delta_{open}(\langle p, q\rangle) = f \Leftrightarrow \eta_p^{\omega\omega} (\rho^2, \rho^2)\text{-realizes } f \wedge \theta_<(q) = dom(f).$$

Nice properties of analytic functions

- Infinitely differentiable
- Cauchy integral formula for derivatives: $f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz$
- Cauchy's theorem: $\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = Z - P$.

These nice analytic properties can be used to obtain nice computability properties. e.g.

Theorem

(Hertling, 1999) *$f \mapsto f'$ is $(\delta_{open}, \delta_{open})$-computable.*

Proof uses Cauchy Integral Formula.

Results on zero-sets Let f range over analytic functions that are not identically zero.

Theorem

Computable Zero-Finding (Matheson, McNicholl 2006) *The map*

$$f \mapsto f^{-1}[\{0\}]$$

is (δ_{open}, ψ)*-computable. Hence, the zero set of a* δ_{open}*-computable analytic function is* ψ*-computable.*

Proof uses Cauchy's Theorem.

Zero sequences

Definition

If u is analytic, then $\{a_n\}_{n=0}^{\infty}$ is a *primary zero sequence* of u if its terms are precisely the zeros of u and the number of times each zero of u is repeated is its multiplicity.

Definition

If u is analytic, then $\{a_n\}_{n=0}^{\infty}$ is a *truncated zero sequence* of u if its terms are precisely the zeros of u *besides zero*, and the number of times each zero of u is repeated is its multiplicity.

Let g range over analytic functions with infinitely many zeros, but not identically zero.

Theorem

Computable primary zero sequence (Matheson, McNicholl 2006) *There is a* $(\delta_{open}, [\rho^2]^\omega)$*-computable function* Ψ *such that* $\Psi(g)$ *is a primary zero sequence of* g *for all* g*.*

Corollary

There is a $(\delta_{open}, [\rho^2]^\omega)$*-computable function* Ψ *such that* $\Psi(g)$ *is a truncated zero sequence of* g *for all* g*.*

Throughout the rest of this talk, we will focus on bounded analytic functions with simply connected domain.

In addition to their analytic properties, these functions have nice *algebraic* properties which can be used to get computability results. In addition, these functions form the mathematical foundation for control theory in engineering.

Outline

The class $H^\infty(\mathbb{D})$

- $H^\infty(\mathbb{D})$ is the set of all bounded analytic functions $f : \mathbb{D} \to \mathbb{C}$.
- For $f \in H^\infty(\mathbb{D})$, let

$$\| f \|_\infty = \sup\{|f(z)| \ : \ z \in \mathbb{D}\}.$$

- $H^\infty(\mathbb{D})$ is a Banach space under $\| \ \|_\infty$.

Definition

$u \in H^\infty(\mathbb{D})$ is *inner* if $\lim_{z \to z_0} |u(z)| = 1$ for almost all $z_0 \in \partial\mathbb{D}$.

Inner functions “generate" $H^\infty(\mathbb{D})$. Two important classes of inner functions:

- Singular functions
- Blaschke products

Definition

A function $s \in H^\infty(\mathbb{D})$ is *singular* if there is a finite positive Borel measure on $\partial\mathbb{D}$, μ, that is singular with respect to Lebesgue measure and such that

$$s(z) = \exp\left\{-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)\right\}$$

Theorem

If s is singular, then s is inner, s has no zeros, and $s(0)$ is a positive real number.

Definition

Let $A = \{a_n\}_{n=0}^{\infty}$ be a sequence of points in $\mathbb{D} - \{0\}$. The product

$$B_{A,k}(z) =_{df} z^k \prod_{n=0}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$$

is called a *Blaschke product*. We abbreviate $B_{A,0}$ with B_A.

Definition

Let $A = \{a_n\}_{n=0}^{\infty}$ be a sequence of points in $\mathbb{D} - \{0\}$. The series

$$\Sigma_A =_{df} \sum_{n=0}^{\infty}(1 - |a_n|)$$

is called the *Blaschke sum* of A. The inequality $\Sigma_A < \infty$. is called the *Blaschke condition*.

Theorem

Let $A = \{a_n\}_{n=0}^{\infty}$ be a sequence of points in $\mathbb{D} - \{0\}$.

1. *If A satisfies the Blaschke condition, then $B_{A,k}$ is an inner function.*
2. *If A satisfies the Blaschke condition, then the terms of A are precisely the zeros of B_A. Furthermore, the number of times a zero of B_A appears in A is its multiplicity.*
3. *If A does not satisfy the Blaschke condition, then $B_A \equiv 0$.*

Importance of Blaschke products

- Factorization
- Estimation of inner functions (Frostman's Theorem)

Theorem

(Factorization of Inner Functions) *If u is an inner function, then there exist unique λ_u, b_u, s_u such that $u = \lambda_u b_u s_u$, $\lambda_u \in \partial\mathbb{D}$, b_u is a (possibly finite) Blaschke product, and s_u is a singular function.*

Capacity For each closed $K \subseteq \mathbb{D}$ and each positive measure σ on K, let $U_\sigma : \mathbb{D} \to \mathbb{D}$ be defined by the equation

$$U_\sigma(z) = \int_K \log \frac{1}{|z - \zeta|} d\sigma(\zeta).$$

Definition

Let $F \subseteq \mathbb{D}$ be closed. We say that F has *zero capacity* if for every positive measure on F, σ, with $\sigma \neq 0$, U_σ is not bounded on any neighborhood of F. Otherwise, we say that F has *positive capacity*. If U is an arbitrary subset of $\mathbb{D}$, then we say that U has positive capacity just in case it has a closed subset with positive capacity; otherwise, we say that it has zero capacity.

Theorem

Every zero-capacity set has measure zero.

However, the Cantor middle-third set has *positive* capacity.

For $a, z \in \overline{\mathbb{D}}$ with $|a| < 1$, let

$$M_a(z) = \frac{z - a}{1 - \overline{a}z}.$$

Theorem

(Frostman's Theorem) *Let u be a non-constant inner function. Then, $M_a \circ u$ is a unit multiple of a Blaschke product for all $a \in \mathbb{D}$ except in a set of capacity zero.*

The set of values of a for which $M_a \circ u$ is not a unit multiple of a Blaschke product is called the *exception set of u*.

As $a \to 0$, $\| M_a \circ u - u \|_\infty \to 0$.

Corollary

If u is a non-constant inner function, and if $\epsilon > 0$, then there is a unit multiple of a Blaschke product B such that $\| u - B \|_\infty < \epsilon$.

Theorem

A truncated zero sequence of an inner function is a Blaschke sequence.

Definition

If u is an inner function, then let Σ_u denote the Blaschke sum of its truncated zero sequence.

If u is an analytic function, then let k_u denote the order of u's zero at 0 if there is one; if $u(0) \neq 0$, then let $k_u = 0$.

Outline

Let g range over analytic functions with infinitely many zeros, but not identically zero.

Theorem

Computable primary zero sequence (Matheson, McNicholl 2006) *There is a* $(\delta_{open}, [\rho^2]^\omega)$*-computable function* Ψ *such that* $\Psi(g)$ *is a primary zero sequence of* g *for all* g*.*

Corollary

There is a $(\delta_{open}, [\rho^2]^\omega)$*-computable function* Ψ *such that* $\Psi(g)$ *is a truncated zero sequence of* g *for all* g*.*

Theorem

Computable Frostman Theorem (McNicholl, 2007) *There is a* $([\rho^2 \to \rho^2]_{\mathbb{D}}, \nu_{\mathbb{N}}, \nu_{\mathbb{N}})$*-computable function* Ψ *such that if* u *is inner (and non-constant) and* $n \in \mathbb{N}$*, then* $M_{\Psi(u,n)} \circ u$ *is a unit multiple of a Blaschke product and* $\| u - M_{\Psi(u,n)} \circ u \|_\infty < 2^{-n}$*.*

This theorem shows we can *effectively* estimate inner functions by unit multiples of Blaschke products.

Computability of k_u Throughout the remaining slides, u ranges over inner functions with infinitely many zeros.

Theorem

(McNicholl, 2007) *$u \mapsto k_u$ is not $([\rho^2 \to \rho^2]_{\mathbb{D}}, \nu_{\mathbb{N}})$-computable.*

Theorem

(McNicholl, 2007) *$(u, \Sigma_u) \mapsto k_u$ is $([\rho^2 \to \rho^2]_{\mathbb{D}}, \rho, \nu_{\mathbb{N}})$-computable.*

Theorem

(Matheson, McNicholl, 2006) *There is a* $[\rho^2]^\omega$*-computable sequence* $A = \{a_n\}_{n=0}^\infty$ *such that* B_A *is not* (ρ^2, ρ^2)*-computable.*

In other words, merely knowing the Blaschke sequence is not enough to compute the Blaschke product.

Theorem

(Matheson, McNicholl 2006) *The map*

$$(A, \Sigma_A) \mapsto B_A$$

is $([\rho^2]^\omega, \rho, [\rho^2 \to \rho^2]_{\mathbb{D}})$*-computable.*

In other words, if you know a Blaschke sequence and its Blaschke sum, then you can compute the Blaschke product.

Theorem

(McNicholl, 2007) *The map* $(A, B_A) \mapsto \sum_A$ *is* $([\rho^2]^\omega, [\rho^2 \to \rho^2]_{\mathbb{D}}, \rho^2)$*-computable. In fact,* $(A, B_A(0)) \mapsto \sum_A$ *is* $([\rho^2]^\omega, \rho, \rho^2)$*-computable.*

In other words, once you know a Blaschke sequence, in order to compute the Blaschke product you have to know the Blaschke sum (or an equivalent piece of information).

Corollary

(**McNicholl 2007)** *Suppose A is $[\rho^2]^\omega$-computable. If B_A maps ρ^2-computable complex numbers to ρ^2-computable complex numbers, then B_A is (ρ^2, ρ^2)-computable.*

This is not the case for power series!

Theorem

(**Caldwell, Pour-El 1975)** *There is a $[\rho]^\omega$-computable sequence $\{a_n\}_{n=0}^\infty$ such that $z \mapsto \sum_{n=0}^\infty a_n z^n$ is (ρ^2, ρ^2)-computable on closed disks but not on the entire plane.*

Theorem

(McNicholl, 2007) *The map* $u \mapsto (\lambda_u, b_u, s_u)$ *is not* $([\rho^2 \rightarrow \rho^2]_{\mathbb{D}}, \rho^2, [\rho^2 \rightarrow \rho^2]_{\mathbb{D}}, [\rho^2 \rightarrow \rho^2]_{\mathbb{D}})$*-continuous.*

In other words, merely knowing an inner function is not enough to compute its factorization.

Theorem

(McNicholl, 2007) *The map* $(u, \sum_u) \mapsto (\lambda_u, b_u, s_u)$ *is* $([\rho^2 \rightarrow \rho^2]_{\mathbb{D}}, \rho, \rho^2, [\rho^2 \rightarrow \rho^2]_{\mathbb{D}}, [\rho^2 \rightarrow \rho^2]_{\mathbb{D}}])$*-computable.*

In other words, for inner functions with infinitely many zeros, knowing the Blaschke sum of u is *sufficient* for computing the factorization of u.

Theorem

(McNicholl, 2007) *The map* $(u, b_u) \mapsto \sum_u$ *is* $([\rho^2 \to \rho^2]_{\mathbb{D}}, [\rho^2 \to \rho^2]_{\mathbb{D}}, \rho)$*-computable.*

In other words, to compute the factorization of u, knowledge of the Blaschke sum of u (or equivalent information) is *necessary*.

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